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இலங்கைப் பரீட்சைத் திணைக்களம்
Department of Examinations, Sri Lanka

අධ්‍යයන පොදු සහතික පත්‍ර (උසස් පෙළ) විභාගය, 2017 අගෝස්තු
கல்விப் பொதுத் தராதரப் பத்திர (உயர் தர)ப் பரீட்சை, 2017 ஓகஸ்ற்
General Certificate of Education (Adv. Level) Examination, August 2017

සංයුක්ත ගණිතය I
இணைந்த கணிதம் I
Combined Mathematics I

10 E I

පැය තුනයි
மூன்று மணித்தியாலம்
Three hours

| Index Number | | | | | | | |
|---|---|---|---|---|---|---|---|

## Instructions:

* *This question paper consists of two parts;*
  **Part A** (Questions 1 - 10) and **Part B** (Questions 11 - 17).
* **Part A:**
  *Answer **all** questions. Write your answers to each question in the space provided. You may use additional sheets if more space is needed.*
* **Part B:**
  *Answer **five** questions only. Write your answers on the sheets provided.*
* *At the end of the time allotted, tie the answer scripts of the two parts together so that* **Part A** *is on top of* **Part B** *and hand them over to the supervisor.*
* *You are permitted to remove **only** **Part B** of the question paper from the Examination Hall.*

**For Examiners' Use only**

| (10) Combined Mathematics I | | |
|---|---|---|
| Part | Question No. | Marks |
| A | 1 | |
| | 2 | |
| | 3 | |
| | 4 | |
| | 5 | |
| | 6 | |
| | 7 | |
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| | 9 | |
| | 10 | |
| B | 11 | |
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| | Total | |
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|---|---|
| Paper I | |
| Paper II | |
| Total | |
| Final Marks | |

**Final Marks**

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**Code Numbers**

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| Marking Examiner | | |
| Checked by: | 1 | |
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| Supervised by: | | |



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## Part A

1. Using the **Principle of Mathematical Induction**, prove that $\sum_{r=1}^{n} r(3r+1) = n(n+1)^2$ for all $n \in \mathbb{Z}^+$.

2. Find all real values of $x$ satisfying the inequality $x^2 - 1 \geq |x+1|$.



[see page three

**3.** Sketch, in an Argand diagram, the locus $l$ of the points that represent complex numbers $z$ satisfying $\text{Arg}\,(z - 2i) = \frac{\pi}{3}$.

Let $P$ and $Q$ be the points in the above Argand diagram that represent the complex numbers $2i$ and $\sqrt{3} + 5i$, respectively. Find the distance $PQ$ and show that the point $Q$ lies on $l$.

**4.** In how many different ways can the eight letters of the word INFINITY be arranged in a row? How many of these arrangements have

(i) all three I letters close to one another?

(ii) exactly one I letter and both N letters as the first three letters?



[see page four

**5.** Let $0 < \alpha < \frac{\pi}{2}$. Show that $\lim_{x \to \alpha} \frac{x^3 - \alpha^3}{\tan x - \tan \alpha} = 3\alpha^2 \cos^2 \alpha$.

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**6.** Let $0 < a < b$. Show that $\frac{\mathrm{d}}{\mathrm{d}x} \sin^{-1}\left(\sqrt{\frac{b-a}{b}} \cos x\right) = -\frac{\sqrt{b-a} \sin x}{\sqrt{a \cos^2 x + b \sin^2 x}}$.

**Hence**, find $\int \frac{\sin x}{\sqrt{a \cos^2 x + b \sin^2 x}} \mathrm{d}x$.

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[see page five

7. A curve $C$ is given parametrically by $x=3\cos\theta-\cos^3\theta$, $y=3\sin\theta-\sin^3\theta$ for $0<\theta<\frac{\pi}{2}$.

Show that $\frac{dy}{dx}=-\cot^3\theta$.

Find the coordinates of the point $P$ on the curve $C$ at which the gradient of the tangent line is $-1$.

8. Let $l_1$ and $l_2$ be the straight lines given by $3x-4y=2$ and $4x-3y=1$ respectively.

(i) Write down the equations of the bisectors of the angles between $l_1$ and $l_2$.

(ii) Find the equation of the bisector of the acute angle between $l_1$ and $l_2$.



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**9.** Let $S$ be the circle given by $x^2 + y^2 - 4 = 0$ and let $l$ be the straight line given by $y = x + 1$. Find the equation of the circle which passes through the points of intersection of $S$ and $l$, and also intersects the circle $S$ orthogonally.

**10.** Show that $\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)^2 = 1 + \sin\theta$ for $-\pi < \theta \leq \pi$. **Hence**, show that $\cos\frac{\pi}{12} + \sin\frac{\pi}{12} = \sqrt{\frac{3}{2}}$ and also find the value of $\cos\frac{\pi}{12} - \sin\frac{\pi}{12}$. Deduce that $\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$.



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கல்விப் பொதுத் தராதரப் பத்திர (உயர் தர)ப் பரீட்சை, 2017 ஓகஸ்ற்
General Certificate of Education (Adv. Level) Examination, August 2017

සංයුක්ත ගණිතය I
இணைந்த கணிதம் I
Combined Mathematics I

10 E I

## Part B

* Answer **five** questions only.

**11.** (*a*) Let $f(x) = 3x^2 + 2ax + b$, where $a, b \in \mathbb{R}$.

It is given that the equation $f(x) = 0$ has two real distinct roots. Show that $a^2 > 3b$.

Let $\alpha$ and $\beta$ be the roots of $f(x)=0$. Write down $\alpha+\beta$ in terms of $a$ and $\alpha\beta$ in terms of $b$.

Show that $|\alpha - \beta| = \frac{2}{3}\sqrt{a^2 - 3b}$.

Show further that the quadratic equation with $|\alpha+\beta|$ and $|\alpha-\beta|$ as its roots is given by $9x^2 - 6\left(|a| + \sqrt{a^2 - 3b}\right)x + 4\sqrt{a^4 - 3a^2 b} = 0$.

(*b*) Let $g(x) = x^3 + px^2 + qx + 1$, where $p, q \in \mathbb{R}$. When $g(x)$ is divided by $(x-1)(x+2)$, the remainder is $3x+2$. Show that the remainder when $g(x)$ is divided by $(x-1)$ is 5, and that the remainder when $g(x)$ is divided by $(x+2)$ is $-4$.

Find the values of $p$ and $q$, and show that $(x+1)$ is a factor of $g(x)$.

**12.** (*a*) Write down the binomial expansion of $(5 + 2x)^{14}$ in ascending powers of $x$.

Let $T_r$ be the term containing $x^r$ in the above expansion for $r = 0, 1, 2, \ldots, 14$.

Show that $\frac{T_{r+1}}{T_r} = \frac{2(14-r)}{5(r+1)}x$ for $x \neq 0$.

**Hence**, find the value of $r$ which gives the largest term of the above expansion, when $x = \frac{4}{3}$.

(*b*) Let $c \geq 0$. Show that $\frac{2}{(r+c)(r+c+2)} = \frac{1}{(r+c)} - \frac{1}{(r+c+2)}$ for $r \in \mathbb{Z}^+$.

**Hence**, show that $\sum_{r=1}^{n} \frac{2}{(r+c)(r+c+2)} = \frac{(3+2c)}{(1+c)(2+c)} - \frac{1}{(n+c+1)} - \frac{1}{(n+c+2)}$ for $n \in \mathbb{Z}^+$.

Deduce that the infinite series $\sum_{r=1}^{\infty} \frac{2}{(r+c)(r+c+2)}$ converges and find its sum.

By using this sum with suitable values for $c$, show that $\sum_{r=1}^{\infty} \frac{1}{r(r+2)} = \frac{1}{3} + \sum_{r=1}^{\infty} \frac{1}{(r+1)(r+3)}$.



[see page eight

13.(*a*) Let $\mathbf{A} = \begin{pmatrix} 2 & a & 3 \\ -1 & b & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & -1 & a \\ 1 & b & 0 \end{pmatrix}$ and $\mathbf{P} = \begin{pmatrix} 4 & 1 \\ 2 & 0 \end{pmatrix}$, where $a, b \in \mathbb{R}$.

It is given that $\mathbf{AB}^{\mathrm{T}} = \mathbf{P}$, where $\mathbf{B}^{\mathrm{T}}$ denotes the transpose of the matrix $\mathbf{B}$. Show that $a = 1$ and $b = -1$, and with these values for $a$ and $b$, find $\mathbf{B}^{\mathrm{T}}\mathbf{A}$.

Write down $\mathbf{P}^{-1}$, and using it, find the matrix $\mathbf{Q}$ such that $\mathbf{PQ} = \mathbf{P}^2 + 2\mathbf{I}$, where $\mathbf{I}$ is the identity matrix of order 2.

(*b*) Sketch in an Argand diagram, the locus $C$ of the points representing complex numbers $z$ satisfying $|z| = 1$.

Let $z_0 = a(\cos\theta + i\sin\theta)$, where $a > 0$ and $0 < \theta < \frac{\pi}{2}$. Find the modulus in terms of $a$ and the principal argument, in terms of $\theta$, of each of the complex numbers $\frac{1}{z_0}$ and $z_0^2$.

Let $P$, $Q$, $R$ and $S$ be the points in the above Argand diagram representing the complex numbers $z_0$, $\frac{1}{z_0}$, $z_0 + \frac{1}{z_0}$ and $z_0^2$, respectively.

Show that when the point $P$ lies on $C$ above,

(i) the points $Q$ and $S$ also lie on $C$, and

(ii) the point $R$ lies on the real axis between 0 and 2.

14.(*a*) Let $f(x) = \dfrac{x^2}{(x-1)(x-2)}$ for $x \neq 1, 2$.

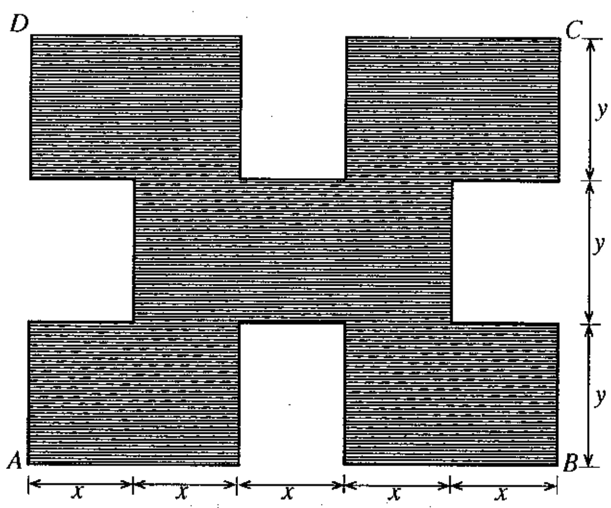
Show that $f'(x)$, the derivative of $f(x)$, is given by $f'(x) = \dfrac{x(4-3x)}{(x-1)^2(x-2)^2}$ for $x \neq 1, 2$.

Sketch the graph of $y = f(x)$ indicating the asymptotes and the turning points.

Using the graph, solve the inequality $\dfrac{x^2}{(x-1)(x-2)} \leq 0$.

(*b*) The shaded region shown in the adjoining figure is of area 385 m². This region is obtained by removing four identical rectangles each of length $y$ metres and width $x$ metres from a rectangle $ABCD$ of length $5x$ metres and width $3y$ metres. Show that $y = \dfrac{35}{x}$ and that the perimeter $P$ of the shaded region, measured in metres, is given by $P = 14x + \dfrac{350}{x}$ for $x > 0$.

Find the value of $x$ such that $P$ is minimum.






[see page nine

**15.**(*a*) (i) Express $\dfrac{1}{x(x+1)^2}$ in partial fractions and **hence**, find $\displaystyle\int \frac{1}{x(x+1)^2}\,dx$.

(ii) Using integration by parts, find $\displaystyle\int xe^{-x}\,dx$ and **hence**, find the area of the region enclosed by the curve $y = xe^{-x}$ and the straight lines $x = 1$, $x = 2$ and $y = 0$.

(*b*) Let $c > 0$ and $I = \displaystyle\int_0^c \frac{\ln(c+x)}{c^2+x^2}\,dx$. Using the substitution $x = c\tan\theta$,

show that $I = \dfrac{\pi}{4c}\ln c + \dfrac{1}{c}J$, where $J = \displaystyle\int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\,d\theta$.

Using the formula $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, where $a$ is a constant, show that $J = \dfrac{\pi}{8}\ln 2$.

Deduce that $I = \dfrac{\pi}{8c}\ln(2c^2)$.

**16.** Let $m \in \mathbb{R}$. Show that the point $P \equiv (0, 1)$ does not lie on the straight line $l$ given by $y = mx$.

Show that the coordinates of any point on the straight line through $P$ perpendicular to $l$ can be written in the form $(-mt,\ t+1)$, where $t$ is a parameter.

**Hence**, show that the coordinates of the point $Q$, the foot of the perpendicular drawn from $P$ to $l$, are given by $\left(\dfrac{m}{1+m^2}, \dfrac{m^2}{1+m^2}\right)$.

Show that, as $m$ varies, the point $Q$ lies on the circle $S$ given by $x^2 + y^2 - y = 0$, and sketch the locus of $Q$ in the $xy$-plane.

Also, show that the point $R \equiv \left(\dfrac{\sqrt{3}}{4}, \dfrac{1}{4}\right)$ lies on $S$.

Find the equation of the circle $S'$ whose centre lies on the $x$-axis, and touches $S$ externally at the point $R$.

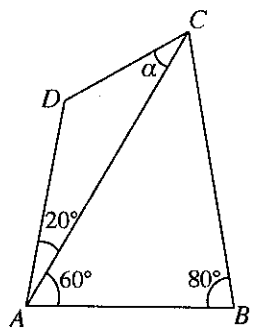
Write down the equation of the circle having the same centre as that of $S'$ and touching $S$ internally.

**17.** (*a*) (i) Show that $\dfrac{2\cos(60° - \theta) - \cos\theta}{\sin\theta} = \sqrt{3}$ for $0° < \theta < 90°$.

(ii) In the quadrilateral $ABCD$ shown in the figure, $AB = AD$, $A\hat{B}C = 80°$, $C\hat{A}D = 20°$ and $B\hat{A}C = 60°$.

Let $A\hat{C}D = \alpha$. Using the Sine Rule for the triangle $ABC$, show that $\dfrac{AC}{AB} = 2\cos 40°$.

Next, using the Sine Rule for triangle $ADC$, show that $\dfrac{AC}{AD} = \dfrac{\sin(20° + \alpha)}{\sin\alpha}$.




Deduce that $\sin(20° + \alpha) = 2\cos 40° \sin\alpha$.

**Hence**, show that $\cot\alpha = \dfrac{2\cos 40° - \cos 20°}{\sin 20°}$.

Now, using the result in (i) above, show that $\alpha = 30°$.



(*b*) Solve the equation $\cos 4x + \sin 4x = \cos 2x + \sin 2x$.

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கல்விப் பொதுத் தராதரப் பத்திர (உயர் தரப்) பரீட்சை, 2017 ஓகஸ்ற்
General Certificate of Education (Adv. Level) Examination, August 2017

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සංයුක්ත ගණිතය II
இணைந்த கணிதம் II
# Combined Mathematics II

**10 E II**

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மூன்று மணித்தியாலம்
***Three hours***

| Index Number | | | | | | | |
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## Instructions:

- *This question paper consists of two parts;*
  **Part A** (Questions 1 – 10) *and* **Part B** (Questions 11 – 17)
- **Part A:**
  *Answer **all** questions. Write your answers to each question in the space provided. You may use additional sheets if more space is needed.*
- **Part B:**
  *Answer **five** questions only. Write your answers on the sheets provided.*
- *At the end of the time allotted, tie the answer scripts of the two parts together so that* **Part A** *is on top of* **Part B** *and hand them over to the supervisor.*
- *You are permitted to remove **only** **Part B** of the question paper from the Examination Hall.*

- In this question paper, $g$ denotes the acceleration due to gravity.

**For Examiners' Use only**

| (10) Combined Mathematics II | | |
|---|---|---|
| **Part** | **Question No.** | **Marks** |
| A | 1 | |
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| Paper II | |
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**Final Marks**

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[see page two

**Part A**

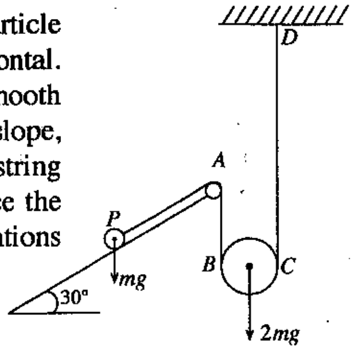
1. A particle $P$ of mass $m$ and a particle $Q$ of mass $\lambda m$ move with speeds $u$ and $v$ respectively, towards each other along the same straight line on a smooth horizontal floor. After their impact, the particle $P$ moves with speed $v$ and the particle $Q$ moves with speed $u$ in opposite directions. Show that $\lambda = 1$, and find the coefficient of restitution between $P$ and $Q$.

2. A balloon, carrying a small uniform ball, starts from rest from a point on the ground at time $t=0$ and moves vertically upwards with uniform acceleration $f$, where $f<g$. At time $t=T$, the ball gets gently detached from the balloon and moves under gravity. Sketch the velocity–time graph for the upward motion of the ball, from $t=0$ until the ball reaches its maximum height. Find the maximum height reached by the ball in terms of $T$, $f$ and $g$.



[see page three

3. In the diagram, $PABCD$ is a light inextensible string attached to a particle of mass $m$ placed on a fixed smooth plane inclined at 30° to the horizontal. The string passes over a fixed small smooth pulley at $A$ and under a smooth pulley of mass $2m$. The point $D$ is fixed. $PA$ is along a line of greatest slope, and $AB$ and $CD$ are vertical. The system is released from rest with the string taut. Show that the magnitude of the acceleration of the particle is twice the magnitude of the acceleration of the movable pulley and write down equations sufficient to determine the tension of the string.




4. A truck of mass $M$ kg is towing a car of mass $m$ kg, along a straight horizontal road, by means of a light inextensible cable which is parallel to the direction of motion of the truck and the car. The resistances to the motion of the truck and the car are $\lambda M$ newtons and $\lambda m$ newtons respectively, where $\lambda (>0)$ is a constant. At a certain instant, the power generated by the engine of the truck is $P$ kW and the speed of the truck and the car is $v \, \mathrm{m\,s^{-1}}$. Show that the tension of the cable at that instant is $\dfrac{1000mP}{(M+m)v}$ newtons.

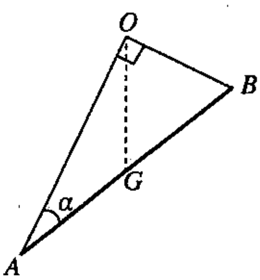


[see page four

**5.** In the usual notation, let $-\mathbf{i}+2\mathbf{j}$ and $2\alpha\mathbf{i}+\alpha\mathbf{j}$ be the position vectors of two points $A$ and $B$ respectively, with respect to a fixed origin $O$, where $\alpha(>0)$ is a constant. Using scalar product, show that $A\hat{O}B = \frac{\pi}{2}$.

Let $C$ be the point such that $OACB$ is a rectangle. If the vector $\overrightarrow{OC}$ lies along the $y$-axis, find the value of $\alpha$.

**6.** A uniform rod $AB$ of length $2a$ and weight $W$, suspended from a fixed point $O$ by two light inextensible strings $OA$ and $OB$, is in equilibrium as shown in the figure. $G$ is the mid-point of $AB$. It is given that $A\hat{O}B = \frac{\pi}{2}$ and $O\hat{A}B = \alpha$. Show that $A\hat{O}G = \alpha$, and find the tensions in the two strings.






[see page five

7. Let $A$ and $B$ be two events of a sample space $\Omega$. In the usual notation, it is given that $P(A \cup B) = \frac{4}{5}$, $P(A' \cup B') = \frac{5}{6}$ and $P(B \mid A) = \frac{1}{4}$. Find $P(A)$ and $P(B)$.

8. A bag contains nine cards. The digit 1 is printed on four of them and the digit 2 is printed on the rest. Cards are drawn from the bag at random, one at a time, without replacement. Find the probability that

(i) the sum of the digits on the first two cards drawn is four,

(ii) the sum of the digits on the first three cards drawn is three.



[see page six

**9.** Values of six observations are $a, a, b, b, x$ and $y$, where $a, b, x$ and $y$ are distinct positive integers with $a<b$. What are the modes of these six observations?

It is given that the sum and the product of these modes are $x$ and $y$ respectively. If the mean of the six observations is $\frac{7}{2}$, find $a$ and $b$.

**10.** The mean and the variance of the ten numbers $x_1, x_2, \ldots, x_{10}$ are 10 and 9 respectively. It is given that the mean of the nine numbers which remain after omitting the number $x_{10}$ is also 10. Find the variance of these nine numbers.





[see page seven

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கல்விப் பொதுத் தராதரப் பத்திர (உயர் தர)ப் பரீட்சை, 2017 ஓகஸ்ற்
General Certificate of Education (Adv. Level) Examination, August 2017

සංයුක්ත ගණිතය II
இணைந்த கணிதம் II
**Combined Mathematics II**

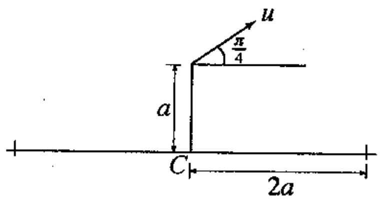
**10 E II**

**Part B**

* Answer **five** questions only.

(In this question paper, $g$ denotes the acceleration due to gravity.)

**11.** (*a*) The base of a vertical tower of height $a$ is at the centre $C$ of a circular pond of radius $2a$, on horizontal ground. A small stone is projected from the top of the tower with speed $u$ at an angle $\frac{\pi}{4}$ above the horizontal. (See the figure.) The stone moves freely under gravity and hits the horizontal plane through $C$ at a distance $R$ from $C$. Show that $R$ is given by the equation $gR^2 - u^2R - u^2a = 0$.



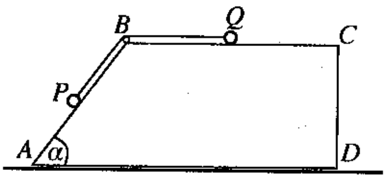

Find $R$ in terms of $u$, $a$ and $g$, and deduce that if $u^2 > \frac{4}{3}ga$, then the stone will not fall into the pond.

(*b*) A ship $S$ is sailing due East with uniform speed $u$ km h$^{-1}$, relative to earth. At the instant when the ship is at a distance $l$ km at an angle $\theta$ South of West from a boat $B$, the boat travels in a straight line path, intending to intercept the ship, with uniform speed $v$ km h$^{-1}$ relative to earth, where $u\sin\theta < v < u$. Assuming that the ship and the boat maintain their speeds and paths, sketch, in the same diagram, the velocity triangles to determine the two possible paths of the boat relative to earth.

Show that the angle between the two possible directions of motion of the boat relative to earth is $\pi - 2\alpha$, where $\alpha = \sin^{-1}\left(\frac{u\sin\theta}{v}\right)$.

Let $t_1$ hours and $t_2$ hours be the times taken by the boat to intercept the ship along these two paths. Show that $t_1 + t_2 = \frac{2lu\cos\theta}{u^2 - v^2}$.

**12.** (*a*) The trapezium $ABCD$ shown in the figure is a vertical cross-section through the centre of gravity of a smooth uniform block of mass $2m$. The lines $AD$ and $BC$ are parallel, and the line $AB$ is a line of greatest slope of the face containing it. Also, $AB = 2a$ and $B\hat{A}D = \alpha$, where $0 < \alpha < \frac{\pi}{2}$ and $\cos\alpha = \frac{3}{5}$. The block is placed with the face containing $AD$ on a smooth horizontal floor.




A light inextensible string of length $l$ $(> 2a)$ passes over a small smooth pulley at $B$, and has a particle $P$ of mass $m$ attached to one end and another particle $Q$ of the same mass $m$ attached to the other end. The system is released from rest with the string taut, the particle $P$ held at the mid-point of $AB$ and the particle $Q$ on $BC$, as shown in the figure.

Show that the acceleration of the block relative to the floor is $\frac{4}{17}g$ and find the acceleration of $P$ relative to the block.

Also, show that the time taken by the particle $P$ to reach $A$ is $\sqrt{\frac{17a}{5g}}$.



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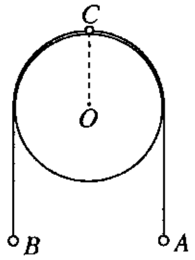
(*b*) Two particles $A$ and $B$, each of mass $m$ are attached to the two ends of a light inextensible string of length $l(>2\pi a)$. A particle $C$ of mass $2m$ is attached to the mid-point of the string. The string is placed over a fixed smooth sphere of centre $O$ and radius $a$ with the particle $C$ at the highest point of the sphere, and the particles $A$ and $B$ hanging freely in a vertical plane through $O$, as shown in the figure. The particle $C$ is given a small displacement on the sphere in the same vertical plane, so that the particle $A$ moves downwards in a straight line path. As long as the particle $C$ is in contact with the sphere, show that $\dot{\theta}^2 = \frac{g}{a}(1-\cos\theta)$, where $\theta$ is the angle through which $OC$ has turned. Show further that the particle $C$ leaves the sphere when $\theta = \frac{\pi}{3}$.




**13.** One end of a light elastic string of natural length $a$ and modulus of elasticity $mg$ is attached to a fixed point $O$ at a height $3a$ above a smooth horizontal floor and the other end is attached to a particle of mass $m$. The particle is placed near $O$ and projected vertically downwards with speed $\sqrt{ga}$. Show that the length of the string $x$ satisfies the equation $\ddot{x}+\frac{g}{a}(x-2a)=0$ for $a \le x < 3a$, and find the centre of this simple harmonic motion.

Using the Principle of Conservation of Energy for the downward motion of the particle until the first impact with the floor, show that $\dot{x}^2=\frac{g}{a}(4ax-x^2)$ for $a \le x < 3a$.

Taking $X=x-2a$, express the last equation in the form $\dot{X}^2 = \frac{g}{a}\left(A^2 - X^2\right)$ for $-a \le X < a$, where $A$ is the amplitude to be determined.

What is the velocity of the particle just before the first impact with the floor?

The coefficient of restitution between the particle and the floor is $\frac{1}{\sqrt{3}}$. For the upward motion of the particle after the first impact, until the string becomes slack, it is given that $\dot{X}^2 = \frac{g}{a}\left(B^2 - X^2\right)$ for $-a \le X < a$, where $B$ is the amplitude of this new simple harmonic motion to be determined.

Show that the total time during which the particle performs downwards and upwards simple harmonic motions described above is $\frac{5\pi}{6}\sqrt{\frac{a}{g}}$.

**14.** (*a*) The position vectors of two distinct points $A$ and $B$ with respect to a fixed origin $O$, **not collinear** with $A$ and $B$, are **a** and **b** respectively. Let $\mathbf{c}=(1-\lambda)\,\mathbf{a}+\lambda\mathbf{b}$ be the position vector of a point $C$ with respect to $O$, where $0<\lambda<1$.

Express the vectors $\overrightarrow{AC}$ and $\overrightarrow{CB}$ in terms of **a**, **b** and $\lambda$.

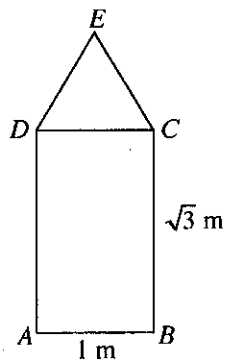
**Hence**, show that the point $C$ lies on the line segment $AB$ and that $AC : CB = \lambda : (1-\lambda)$.

Now, suppose that the line $OC$ bisects the angle $AOB$. Show that $|\mathbf{b}|(\mathbf{a}\cdot\mathbf{c}) = |\mathbf{a}|(\mathbf{b}\cdot\mathbf{c})$ and **hence**, find $\lambda$.

(*b*) In the figure, $ABCD$ is a rectangle with $AB = 1$ m and $BC = \sqrt{3}$ m, and $CDE$ is an equilateral triangle. Forces of magnitude 5, $2\sqrt{3}$, 3, $4\sqrt{3}$, $P$ and $Q$ newtons act along $BA$, $DA$, $DC$, $CB$, $CE$ and $DE$ respectively, in the directions indicated by the order of the letters. This system of forces reduces to a couple. Show that $P=4$ and $Q=8$, and find the moment of this couple.

Now, the directions of forces acting along $BA$ and $DA$ are reversed, but their magnitudes remain the same. Show that the new system reduces to a single resultant force of magnitude $2\sqrt{37}$ newtons.

Show further that the distance from $A$ to the point at which the line of action of this resultant force meets $BA$ produced is $\frac{7}{4}$ m.



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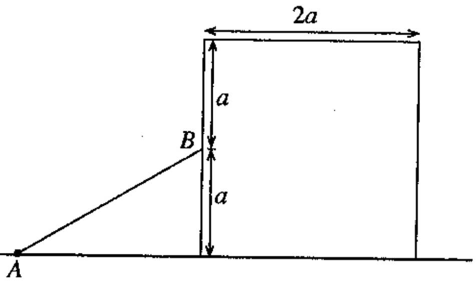
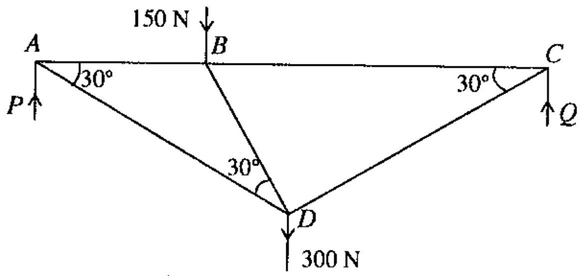
**15.** (*a*) A uniform cubical block of weight $W$ and side of length $2a$ is placed on a rough horizontal floor. A uniform rod $AB$ of weight $2W$ and length $2a$ has its end $A$ smoothly hinged to a point on the horizontal floor and has the end $B$ against a smooth vertical face of the cube at its centre. The vertical plane through the rod is perpendicular to that vertical face of the block and the system stays in equilibrium. (See the figure for the relevant vertical cross-section.) The coefficient of friction between the cubical block and the rough horizontal floor is $\mu$. Show that $\mu \geq \sqrt{3}$.



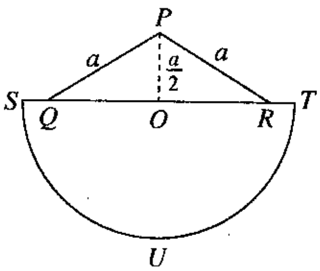

(*b*) The figure shows a framework consisting of five light rods $AB$, $BC$, $AD$, $BD$ and $CD$ freely jointed at their ends. $AB = a$ metres and $BC = 2a$ metres, and $B\hat{A}D = B\hat{D}A = B\hat{C}D = 30°$. The framework is loaded with weights 150 N at $B$ and 300 N at $D$. It is in equilibrium in a vertical plane supported by two vertical forces $P$ and $Q$ at $A$ and $C$ respectively, so that $AB$ and $BC$ are horizontal. Show that $P = 250$ N.




Draw a stress diagram using Bow's notation and **hence**, find the stresses in all the rods and state whether they are tensions or thrusts.

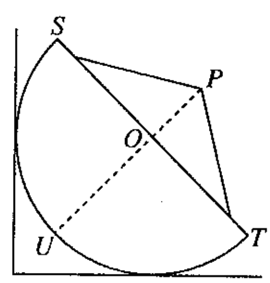
**16.** Show that the centre of mass of a thin uniform wire in the shape of a semi-circular arc of centre $C$ and radius $a$, is at a distance $\frac{2a}{\pi}$ from $C$.

In the adjoining figure, $PQ$, $PR$ and $ST$ are three straight line pieces cut from a thin uniform wire of mass $\rho$ per unit length. The two pieces $PQ$ and $PR$ are welded to each other at the point $P$ and then welded to $ST$ at the points $Q$ and $R$. It is given that $PQ = PR = a$, $ST = 2a$ and $PO = \frac{a}{2}$, where $O$ is the mid-point of both $QR$ and $ST$. Also, $SUT$ is a semicircular arc of centre $O$ and radius $a$ made up of a thin uniform wire of mass $k\rho$ per unit length, where $k(>0)$ is a constant. The rigid plane wire-frame $L$ shown in the figure has been made by welding the semicircular wire $SUT$ to the wire $ST$ in the plane of $PQR$ at the points $S$ and $T$. Show that the centre of mass of $L$ is at a distance $\left(\frac{\pi k + 4k + 3}{\pi k + 4}\right)\frac{a}{2}$ from $P$.




The wire frame $L$ is in equilibrium with its circular part touching a smooth vertical wall and a horizontal ground rough enough to prevent slipping, and its plane perpendicular to the wall as shown in the adjoining figure. Mark the forces acting on $L$ and show that $k > \frac{1}{4}$.

Now, let $k = 1$. The equilibrium is maintained in the above position even after a particle of mass $m$ is attached to $L$ at the point $P$. Show that $m < 3\rho a$.



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17. (*a*) Each of the bags $A$, $B$ and $C$ contains only white balls and black balls which are identical in all respects, except for colour. The bag $A$ contains 4 white balls and 2 black balls, the bag $B$ contains 2 white balls and 4 black balls, and the bag $C$ contains $m$ white balls and $(m+1)$ black balls. A bag is chosen at random and two balls are drawn from that bag at random, one after the other, without replacement. The probability that the first ball drawn is white and the second ball drawn is black, is $\frac{5}{18}$. Find the value of $m$.

Also, find the probability that the bag $C$ was chosen, given that the first ball drawn is white and the second ball drawn is black.

(*b*) The following table gives the distribution of marks obtained by a group of 100 students for their answers to a Statistics question:

| Marks range | Number of students |
|---|---|
| 0–2 | 15 |
| 2–4 | 25 |
| 4–6 | 40 |
| 6–8 | 15 |
| 8–10 | 5 |

Estimate the mean $\mu$ and the standard deviation $\sigma$ of this distribution.

Also, estimate the coefficient of skewness $\kappa$ defined by $\kappa = \frac{3(\mu - M)}{\sigma}$, where $M$ is the median of the distribution.

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