G.C.E. (A. L.) Support Seminar - 2016

| Combined Mathematics I | *Three hours* |

Instructions :

* Answer all questions in **Part A**
* Answer only **five** questions in **Part B.**

Part A

1. By using the Principle of Mathematical Induction, prove that the expression $4^n + 15n - 1$ is divisible by 9 for all positive integers n.

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2. Write the rth term T_r in the expansion of $\left(\sqrt{2} + 7^{\frac{1}{5}}\right)^{10}$ and find the sum of the rational terms in the expansion.

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3. How many different ways are there of forming a group of 5 children from 8 boys and 6 girls such that both sexes are represented in the group?

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4. By using the Argand diagram, obtain the point that represents the complex number $Z = Z_0$ such that $\text{Arg } Z = \dfrac{\pi}{4}$ and $\text{Arg }(Z - 3) = \dfrac{\pi}{2}$, and find $\text{Arg}(Z_0 - 6i)$.

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5. Find the value of the real non - zero constant k such that $\displaystyle\lim_{x\to 0} \frac{(1+kx)^2 - (1-kx)^2}{\sqrt{1+k^2x} - \sqrt{1-k^2x}} = 1$.

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6. Sketch the graphs of the functions $y = (x-2)^2$ and $y = 4 - 2x$ in the same coordinate plane. Show that the area bounded by these graphs is $\frac{4}{3}$ square units.

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7. A curve is defined by the equations $x = t^2$ and $y = at^3 - t^2$, where $a \in \mathbb{R}^+$. If the tangents drawn to the curve at the points given by $t = 1$ and $t = -1$ are perpendicular to each other, find the value of a.

8. Let $A = (2, -1)$ and $B = (4, -3)$. The point $C = (3t, -t)$, $t \in \mathbb{R}$ lies on the perpendicular bisector of AB. Find the value of t and then find the coordinates of the point D such that $ACBD$ is a rhombus.

9. Find the equations of the circles that touch the coordinate axes and the straight line $3x - 4y + 12 = 0$.

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10. Show that $\cot \alpha - \tan \alpha = 2 \cot 2\alpha$.

Hence show that $\tan \alpha + 2 \tan 2\alpha + 4 \tan 4\alpha + 8 \cot 8\alpha = \cot \alpha$.

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Part B

11. (a) In terms of a, b and c, find the condition that needs to be satisfied for the quadratic equation $ax^2 + bx + c = 0$ to have coincident roots.

Express the equation $\dfrac{a}{x+c} + \dfrac{b}{x-c} = \dfrac{k}{2x}$ in quadratic form. Show that if k_1 and k_2 are the values of k for which the equation has coincident roots, then $|k_1 - k_2| = 4\sqrt{ab}$. Here $c \neq 0$.

(b) Let $f(x) = x^2 + 6x + 20 + \lambda(x^2 - 3x - 12)$, $x \in \mathbb{R}$. Here λ is a parameter.

 (i) Find the value of λ such that $f(x)$ is a linear polynomial.

(ii) Find the value of λ if the equation $f(x) = 0$ has two roots equal in magnitude but of opposite signs.

(iii) Assuming that $f(x)$ is a quadratic function, find a, b and h in terms of λ, such that $f(x) = h - b(x - a)^2$. If the maximum value of $f(x)$ occurs at $x = 2$, find the value of λ and the maximum value of $f(x)$.

12. (a) The set of real values of x that satisfy the inequality $|l - 5x| < |kx - 3|$, where l and k are real positive constants is $\{x \mid -2 < x < 4/9\}$. Sketch the graphs of $y = |l - 5x|$ and $y = |kx - 3|$ on the same figure and find the values of l and k.

(b) The sum of the first n terms of the infinite series $\displaystyle\sum_{r=1}^{\infty} U_r$ is $\dfrac{3n}{2n+1}$ where $n = 1, 2, 3 ....$ Show that this infinite series is convergent and find its sum. Determine U_r and hence show that the sum of the first n terms of the infinite series $\displaystyle\sum_{r=1}^{\infty} r^2 U_r$ is $\dfrac{3n(n+1)}{2(2n+1)}$. Does this series converge? Justify your answer.

13. (a) Let $A = \begin{bmatrix} 3 & p \\ -2 & -3 \end{bmatrix}$, where $p \in \mathbb{R}$. Find the non-zero values of p for which the matrix A has an inverse. Find the value of the constant p for which $A^{-1} = A$, and hence obtain two non-zero matrices B and C of order 2 such that $BC = 0$. Here, 0 is the zero matrix of order 2.

(b) The conjugate of the complex number Z is denoted by $\overline{Z}$.
 (i) Show that $Z\overline{Z} = |Z|^2$.
(ii) Z_1 and Z_2 are two complex numbers. Show that $\overline{Z_1 Z_2} = \overline{Z}_1 \, \overline{Z}_2$.
(iii) If $\left| \dfrac{\overline{Z}_1 - 2\overline{Z}_2}{2 - Z_1 \overline{Z}_2} \right| = 1$ for the two complex numbers Z_1 and Z_2, and $|Z_2| \neq 1$, show that $|Z_1| = 2$.

(c) The complex number Z is represented in the Argand diagram by the point P. Plot the locus of P such that $|Z - 3| < 2$ and $\mathrm{Arg}(Z - 3) = \dfrac{\pi}{3}$.

14. (a) Consider $y = (\sin x)^x$ where $0 < x < \pi$. Find $\dfrac{dy}{dx}$.

(b) It has been decided that a tank should be of volume 45π cubic units, and that while it should be in the shape of a right circular cylinder with a bottom, having a hemispherical lid protruding outward from the open end of the cylinder, the surface area of the tank should be minimum. By taking the height of the cylinder to be y units and the base radius to be x units, show that the outer surface area of the tank is $A = 3\pi x^2 + 2\pi xy$ square units and that $y = \dfrac{45}{x^2} - \dfrac{2x}{3}$ units. Find the value of y for which A is minimum.

(c) Let $f(x) = \dfrac{a}{(x-1)^2} + \dfrac{b}{(x+1)}$, where $x \ne \pm 1$ and a, b are real constants. If it is given that $(0, 2)$ is a turning point of the graph of $y = f(x)$, determine a and b and show that $(0, 2)$ is the only turning point. Sketch the graph of $f(x)$, clearly indicating the turning point and the asymptotes. Hence sketch the graph of $|f|(x)$.

15. (a) By using a suitable substitution, show that $\displaystyle\int_0^1 \dfrac{1}{(2+x)^{1/2}\,(2-x)^{3/2}}\ dx = \dfrac{\sqrt{3}-1}{2}$.

(b) Express $G(x) = \dfrac{1}{(x+2)\,(x^2+8)}$ in terms of partial fractions.

Deduce that $g(x) = \dfrac{1}{12\sqrt{2}}\tan^{-1}\left(\dfrac{x}{2\sqrt{2}}\right) + \dfrac{1}{24}\ln\left[\dfrac{(x+2)^2}{x^2+8}\right] + C$ where C is an arbitrary constant and g is a function such that $g'(x) = G(x)$.

(c) If $I_n = \displaystyle\int x^n \sin x\ dx,$ show that $I_n + n(n-1)I_{n-2} = x^{n-1}(n\sin x - x\cos x)$. Here $n \in \mathbb{Z}^+$.

16. (a) Find the equations of the bisectors of the angles between the two non-parallel straight lines given by $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$.
Two sides of a square lie on the bisectors of the angles between the straight lines $4x + y + 3 = 0$ and $x + 4y - 3 = 0$. One of the vertices of the square is $(0, 2)$. Show that there are two such squares and find the equations of the sides of the two squares.

(b) The circle $S^1 \equiv x^2 + y^2 - 2x + 4y - 3 = 0$ bisects the circle $S = 0$. If the point $(1, 1)$ lies on $S = 0$, show that the locus of the centre of this circle is also a circle, and find its centre and radius.

17. (a) Show that $a = b \cos C + c \cos B$ in the usual notation, for any triangle ABC. Hence deduce that $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$.

If a, b, c are in an arithmetic progression, deduce that $\cos A + \cos C + 2\cos B = 2$ and that $\cos \dfrac{A - C}{2} = 2 \sin \dfrac{B}{2}$.

(b) If $0 < x,\ y < \dfrac{\pi}{2}$ and $\sin x > \cos y$, prove that $x + y > \dfrac{\pi}{2}$.

(c) Express $f(x) = 3 \cos^2 x + 8 \sin x \cos x - 3 \sin^2 x$ in the form $f(x) = A \sin(2x + \alpha)$. Here $A > 0$ and α is an acute angle. Hence find the general solution of $f(x) = \dfrac{5}{2}$. Sketch the graph of $y = f(x)$ for $-\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$.

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G.C.E. (A. L.) Support Seminar - 2016

Combined Mathematics II	*Three hours*

Instructions :

* Answer all questions in **Part A**
* Answer only **five** questions in **Part B.**

Part A

1. Two particles P and Q of mass $2m$ each are attached to the two ends of a light inelastic string which passes over a smooth fixed pulley. The system hangs freely in equilibrium under gravity, with the string taut and the portions not in contact with the pulley vertical. Another particle of mass m which is released from rest from a height h above P, collides with P and coalesces, and the composite body commences its motion. Find the initial velocity of the system, and the impulsive tension in the string due to the jerk.

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2. A pump raises water to a height of 4 m and ejects it at a rate of 8 ms^{-1}. If the cross-sectional area of the pipe through which the water flows is 0.005 m^2, find the power of the pump.

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3. A particle is projected under gravity, in a vertical plane, from a point O, with velocity u, at an angle θ to the horizontal. After time T, it moves in a direction perpendicular to the initial direction of the projection. By using the equation $\mathbf{v} = \mathbf{u} + \mathbf{a}t$ and drawing a vector triangle, show that $T = \dfrac{u}{g}\operatorname{cosec}\theta$.

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4. Two smooth identical elastic spheres of masses m and km, move with velocities $5u$ and u respectively in opposite directions, on a smooth horizontal plane, and collide directly. After the collision, the velocities of the two spheres are equal in magnitude to exactly half the initial velocities, but in the opposite directions. Find the value of k, the coefficient of restitution between the two spheres, and the impulse which acted on each sphere due to the collision.

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5. $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = \lambda\mathbf{i} + \mu\mathbf{j}$. Here, λ, μ are real constants such that $\mu > 0$, and $\mathbf{i}, \mathbf{j}$ are the usual unit vectors. If $|\boldsymbol{b}| = 1$ and the two vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular to each other, find the constants λ and μ.

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6. Find the minimum force that is required to move an object of weight w upwards slightly, along the line of greatest slope of a rough plane inclined at an angle α to the horizontal. The angle of friction between the object and the plane is λ.

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7. The probability of two children hitting a target are $\frac{1}{3}$ and $\frac{1}{4}$ respectively. Each child is allowed just one chance to shoot at the target.
   (i) Find the probability of exactly one child hitting the target.
   (ii) If exactly one child hits the target, find the probability of it being the first child.

8. If the two events A and B satisfy the following conditions, find $P(A|B)$.
   (i) The probability of A occurring and B not occurring is 0.2.
   (ii) The probability of B occurring and A not occurring is 0.1.
   (iii) The probability of neither A nor B occurring is 0.6.

9. The mean and the standard deviation of the set of observations {2, 3, 4, 5, 6, 7, 8} are 5 and 2 respectively. Using this, find the mean and standard deviation of each of the following sets of observations.

    (i)   {12, 13, 14, 15, 16, 17, 18}

    (ii) {20, 30, 40, 50, 60, 70, 80}

    (iii) {2a + b, 3a + b, 4a + b, 5a + b, 6a + b, 7a + b, 8a + b}; Here a and b are constants.

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10. Let $u_i = \dfrac{x_i - 35}{a}$ where a is a constant. The mean value of the distribution given by x is 33. The following table gives the frequency distribution of u.

u_i	-3	-2	-1	0	1	2
f_i	5	10	25	30	20	10

Find the class intervals of the distribution given by x and write down the distribution.

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Part B

11. (a) Two objects P and Q are projected under gravity from the points A and B with position vectors $-2\mathbf{i} - 2\mathbf{j}$ and $2\mathbf{i} + 8\mathbf{j}$ respectively, with initial velocities $\mathbf{i} + 25\mathbf{j}$ and $a\mathbf{i} + 20\mathbf{j}$ respectively. Draw velocity time graphs for their horizontal and vertical motions and hence, if they collide, find the time it takes for them to collide and the value of a.

(b) A ship travels towards the East with uniform speed u. An airplane which leaves the ship, travels to an island and immediately returns to the ship. Throughout the journey, the airplane flies along a horizontal line with uniform velocity v, in a direction which makes an angle α East of South relative to the ship. Draw the velocity triangles for the journey out and the return journey of the airplane in the same figure.

When observed from the island, if the airplane changes its direction by an angle of $\frac{\pi}{2}$ during its flight, show that $v = u$ and that the total time taken for the journey is $\frac{2d}{u \cos \alpha}$. Here d is the shortest distance between the ship's path and the island.

12. (a) A particle of mass m which is placed at the lowest point inside a fixed smooth hollow sphere of internal radius a, is projected with a horizontal velocity u. The particle moves in a vertical plane through the centre O of the sphere. Here $u^2 > 2ag$. If this particle leaves the surface of the sphere at the point A, and if the acute angle that OA makes with the upward vertical is α, show that $\cos \alpha = \dfrac{u^2 - 2ag}{3ag}$, and hence show that $u^2 < 5ag$.

Furthermore, if it is given that $\cos \alpha = \dfrac{1}{\sqrt{3}}$ show that the particle passes through O.

(b)
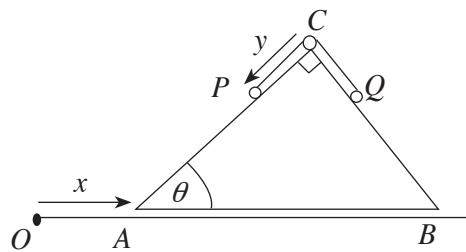
The wedge of mass $5m$ shown in the above figure, with vertical cross section the right triangle ACB with $\tan \theta = \dfrac{4}{3}$, is placed such that AB is in contact with a smooth horizontal plane. Two particles P and Q, each of mass m, which are connected by a light inextensible string which passes over a small light smooth pulley fixed at C, are placed on the smooth surfaces of the wedge along AC and BC, and the system is released from rest. After time t, if the distance of the wedge from the fixed point O is x, and the distance from C to the particle P is y, by applying the law of conservation of momentum and the law of conservation of energy, show that,

(i) $\dot{y} = 5\dot{x}$

(ii) $35\dot{x}^2 + 10\dot{y}^2 - 14\dot{x}\dot{y} - 2gy = $ a constant.

Hence show that the acceleration of the wedge is $\dfrac{1}{43}g$ and the tension in the string is $\dfrac{30}{43}mg$.

13. A particle of mass m is attached to one end of a light elastic string of natural length a and modulus of elasticity $2mg$. The other end of the string is attached to a fixed point O on a smooth plane inclined at an angle of 30° to the horizontal. The particle lies on the plane with the string lying along the line of greatest slope of the plane. The particle is pulled along the line of greatest slope through O, until the length of the string is $2a$, and released from rest. Let x be the extension of the string at time t. Show that for the motion of the particle, the extension x satisfies the equation $\ddot{x} = -\dfrac{2g}{a}\left(x - \dfrac{a}{4}\right)$. By assuming that the solution to this equation is of the form $x = \dfrac{a}{4} + A\cos\omega t + B\sin\omega t$, find the values of the constants A, B, ω. Hence find the centre and the amplitude when the particle is in simple harmonic motion. Show that the speed of the particle when it first arrives at the position where the string attains its natural length is $\sqrt{ag}$. Show that the particle reaches the point O, after a time period $\left[\pi - \cos^{-1}\left(\dfrac{1}{3}\right) + 2\sqrt{2}\,\right]\sqrt{\dfrac{a}{2g}}$, from the moment of release. Find the tension in the string when it is at its maximum length.

14. (a) **a, b, c** are unit vectors.

   (i) If the vectors $\mathbf{a} + 2\mathbf{b}$ and $5\mathbf{a} - 4\mathbf{b}$ are perpendicular to each other, find the angle between the two vectors **a** and **b**.

   (ii) Show that $\left|a - b\right|^2 + \left|b - c\right|^2 + \left|c - a\right|^2 \leq 9$

  (b) *ABCD* is a square of side length 2m. Forces of magnitude 3N, 2N, 4N, 8N, $5\sqrt{2}$ N and $2\sqrt{2}$ N act along $\overrightarrow{BA}$, $\overrightarrow{BC}$, $\overrightarrow{DC}$, $\overrightarrow{DA}$, $\overrightarrow{AC}$ and $\overrightarrow{BD}$ respectively.

   (i) Find the magnitude, direction and line of action of the force that should be introduced to this system for it to be in equilibrium.

   (ii) Find the magnitude, direction and line of action of the force that should be introduced to this system to reduce it to a couple of magnitude 39Nm acting in the sense *ABC*.

   (iii) Find the magnitude and sense of the couple that should be introduced to reduce this system of forces to a single force acting at *B*.

15. (a) *AB* and *BC* are two uniform rods of equal length and equal weight W. The two rods have been smoothly jointed at *B* and the system is kept in equilibrium with *A* hinged to a point on a vertical wall and *C* below *B*, by a horizontal force W acting at *C* in a direction outward from the wall.
Show the following.

   (i) The angle that *BC* makes with the downward vertical is $\tan^{-1} 2$.

   (ii) The reaction at *B* is $\sqrt{2}\,\text{W}$, and it acts at an angle of 45° to the horizontal.

   (iii) The angle that *AB* makes with the downward vertical is $\tan^{-1}\left(\dfrac{2}{3}\right)$.

   (iv) The reaction at *A* is $\sqrt{5}\,\text{W}$, and it acts at an angle of $\tan^{-1}(2)$ to the horizontal.

  (b)
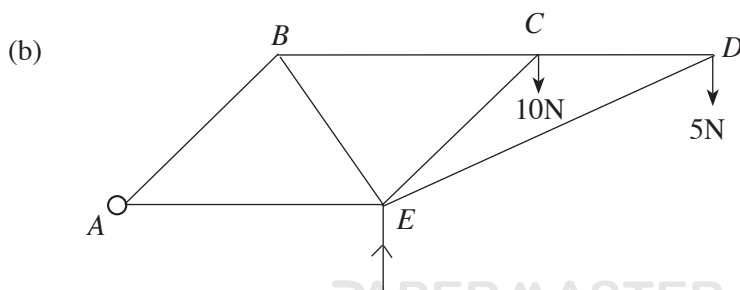
The framework *ABCDE* has been formed by smoothly joining 7 light rods. Here *AB = AE = BE = BC = CE = CD*. This framework which has been smoothly hinged at *A* is kept in equilibrium in a vertical plane, with *AE* and *BCD* horizontal, by a vertical force acting at *E*. Loads of 5N and 10N hang at the joints *D* and *C* respectively.

(i)   Find the vertical force acting at *E*.

(ii)   Find the vertical and horizontal components of the reaction at the hinge *A*.

(iii)   By using Bow's notation, find the stress in each rod and determine whether it is a tension or a thrust.

16.   Find the location of the centre of mass of a solid uniform hemisphere of centre *O*, radius *a* and mass *w*.

(a)   Hence show that, when a solid hemisphere of radius *k*a ($k < 1$) and centre *O* is removed from the above hemisphere, the distance of the centre of mass of the remaining portion from *O* is
$$\frac{3a \, (1 + k^2) \, (1 + k)}{8 \, (1 + k + k^2)} \, .$$

(b)   A composite object is made by joining the hemispherical portion that was removed to the remaining portion, as shown in the figure.
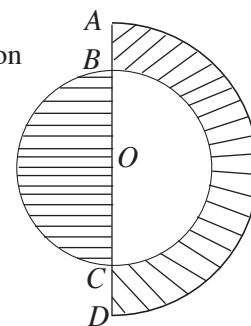
    (i)   Show that the mass of the portion that was removed is a fraction of k^3 of the total mass.

    (ii)   Find the distance from *O* to the centre of mass of the composite object.

    (iii)   Show that for the composite object to be in equilibrium with any point on the curved surface of the larger hemispherical portion in contact with a smooth horizontal table, $k^2 = \frac{1}{\sqrt{2}}$.

17. (a)   In analyzing the records on the boys who receive treatment at a children's the hospital, probabilities related to the following events *A, B, C* were discovered for a boy picked at random from those who receive treatment.

A :   The child has the wheeze

B :   The child has a headache

C :   The child has a stomach ache

It is given that the events *A, B* and *C* are mutuallay independent and that $P(A) = 0.1$, $P(A \cup B) = 0.37$ and $P(C) = 0.2$.

  (i)   Show that $P(B) = 0.3$

  (ii)   Find $P(B' \,|\, A')$. Here A' and B' are the complements of *A* and *B* respectively.

  (iii)   Find the probability of a child having a stomach ache but neither the wheeze nor a headache.

  (iv)   If it is given that the child is suffering from exactly one of the above mentioned conditions, find the probability of it being the wheeze.

(b) (i) Define the mean $\bar{x}$ of the set of values $x_1, x_2, \ldots, x_n$.

A and B record the marks scored for mathematics by 28 students at a term test.

A records the following marks for 7 students.

28, 56, 23, 94, 08, 05, 13

B then records the marks for the remaining 21 students. The sum of these marks is 846.

(α)   Calculate the mean value of the marks of the 28 students.

A then realized that the number 94 should have been 49 and the number 05 should have been 50. A corrects these figures.

(β)   State giving your reasons, the effects this will have on the mean.

(ii) Define the standard deviation s of the set of values $x_1, x_2, \ldots, x_n$ and show that $s^2 = \dfrac{\sum_{r=1}^{n} x_r^2}{n} - \bar{x}^2$.

Let X and Y be two disjoint sets of numbers containing 20 and 10 elements respectively.

The sum of the numbers in X is 320 and the sum of their squares is 5840. The sum of the numbers in Y is 130 and the sum of their squares is 2380. Calculate the mean and the standard deviation of X and $X \cup Y$.

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