Part A

1. When $n = 1$, L.H.S $= \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$

    R.H.S $= \begin{pmatrix} 1+2\times1 & -4\times1 \\ 1 & 1-2\times1 \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} = $ L.H.S

    $\therefore$ The result is true for $n = 1$. ⑤

    Assume that the result is true for $n = p$.

    Then, $\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^p = \begin{pmatrix} 1+2p & -4p \\ p & 1-2p \end{pmatrix}$ ⑤

    $\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^{p+1} = \begin{pmatrix} 1+2p & -4p \\ p & 1-2p \end{pmatrix}\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$ ⑤

    $= \begin{pmatrix} 3(1+2p)-4p & -4(1+2p)+4p \\ 3p+1-2p & -4p-(1-2p) \end{pmatrix}$

    $= \begin{pmatrix} 3+2p & -4-4p \\ 1+p & -1-2p \end{pmatrix}$

    $= \begin{pmatrix} 1+2(p+1) & -4(p+1) \\ (p+1) & 1-2(p+1) \end{pmatrix}$ ⑤

    $\therefore$ The result is true for $n = p + 1$.

    $\therefore$ By the Principle of Mathematical Induction, the result is true for all positive integers n. ⑤

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2. $\left(\sqrt{5}+\sqrt{3}\right)^4 + \left(\sqrt{5}-\sqrt{3}\right)^4$

    $= {}^4C_0\left(\sqrt{5}\right)^4 + {}^4C_1\left(\sqrt{5}\right)^3.\sqrt{3} + {}^4C_2\left(\sqrt{5}\right)^2\left(\sqrt{3}\right)^2 + {}^4C_3\left(\sqrt{5}\right)\left(\sqrt{3}\right)^3 + {}^4C_4\left(\sqrt{3}\right)^4 + {}^4C_0\left(\sqrt{5}\right)^4$ ⑤
    $\quad - {}^4C_1\left(\sqrt{5}\right)^3\left(\sqrt{3}\right) + {}^4C_2\left(\sqrt{5}\right)^2\left(\sqrt{3}\right)^2 - {}^4C_3\left(\sqrt{5}\right)\left(\sqrt{3}\right)^3 + {}^4C_4\left(\sqrt{3}\right)^4$

    $= 2\left[{}^4C_0\left(\sqrt{5}\right)^4 + {}^4C_2\left(\sqrt{5}\right)^2\left(\sqrt{3}\right)^2 + {}^4C_4\left(\sqrt{3}\right)^4\right]$

    $= 2[25 + 6\times15 + 9]$ ⑤
    $= 2 \times 124$
    $= 248$ ——— (1)

$$\left(\sqrt{5}+\sqrt{3}\right)\left(\sqrt{5}-\sqrt{3}\right) \quad = \quad 5-3=2 \quad ⑤$$

$$\Rightarrow \left(\sqrt{5}-\sqrt{3}\right) = \frac{2}{\sqrt{5}+\sqrt{3}}$$

$$\Rightarrow 0 < \left(\sqrt{5}-\sqrt{3}\right) < 1 \quad ⑤ \quad \left(\because \sqrt{5}+\sqrt{3} > 2\right)$$

$$\Rightarrow 0 < \left(\sqrt{5}-\sqrt{3}\right)^4 < 1$$

$$\therefore \text{From (1), } 0 < 248 - \left(\sqrt{5}+\sqrt{3}\right)^4 < 1$$

$$\Rightarrow 247 < \left(\sqrt{5}+\sqrt{3}\right)^4 < 248$$

$$\Rightarrow n = 247 \qquad ⑤ \qquad \boxed{25}$$

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3. $(3-2i)(7-5i)$ $\quad = \quad 21+10i^2-14i-15i$ $\qquad ⑤$

$\qquad\qquad\qquad\quad = \quad 11-29i \quad$ (since $i^2=-1$) $\qquad ⑤$

$\therefore 11+29i \qquad = \quad (3+2i)(7+5i)$ $\qquad ⑤$

$\quad 11^2+29^2 \qquad = \quad 11^2-(29i)^2$

$\qquad\qquad\qquad\quad = \quad (11-29i)(11+29i)$

$\qquad\qquad\qquad\quad = \quad (3-2i)(7-5i)\,(3+2i)(7+5i)$

$\qquad\qquad\qquad\quad = \quad (9-4i^2)(49-25i^2)$ $\qquad ⑤$

$\qquad\qquad\qquad\quad = \quad (9+4)(49+25)$

$\qquad\qquad\qquad\quad = \quad 13 \times 74$ $\qquad ⑤ \qquad \boxed{25}$

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4. $\displaystyle\lim_{x\to\frac{\pi}{2}} \frac{(2x-\pi)\cos x}{2\cos^2 x - \left(\frac{\pi}{2}-x\right)^2 \sin x}$

$$= \lim_{\left(x-\frac{\pi}{2}\right)\to 0} \frac{2\left(x-\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}-x\right)}{2\sin^2\left(\frac{\pi}{2}-x\right)-\left(\frac{\pi}{2}-x\right)^2\cos\left(\frac{\pi}{2}-x\right)}$$

⑤

$$= \lim_{\left(x-\frac{\pi}{2}\right)\to 0} \frac{-2\sin\left(x-\frac{\pi}{2}\right)\Big/\left(x-\frac{\pi}{2}\right)}{\dfrac{2\sin^2\left(x-\frac{\pi}{2}\right)}{\left(x-\frac{\pi}{2}\right)^2}-\cos\left(x-\frac{\pi}{2}\right)}$$

⑤

$$= \lim_{\left(x-\frac{\pi}{2}\right)\to 0} \frac{-2\dfrac{\sin\left(x-\frac{\pi}{2}\right)}{\left(x-\frac{\pi}{2}\right)}}{2\left[\dfrac{\sin\left(x-\frac{\pi}{2}\right)}{\left(x-\frac{\pi}{2}\right)}\right]^2 - \cos\left(x-\frac{\pi}{2}\right)}$$

$$= \frac{-2\left[\lim_{\left(x-\frac{\pi}{2}\right)\to 0}\dfrac{\sin\left(x-\frac{\pi}{2}\right)}{\left(x-\frac{\pi}{2}\right)}\right] \;⑤}{2\left[\lim_{\left(x-\frac{\pi}{2}\right)\to 0}\dfrac{\sin\left(x-\frac{\pi}{2}\right)}{\left(x-\frac{\pi}{2}\right)}\right]^2 - \lim_{\left(x-\frac{\pi}{2}\right)\to 0}\cos\left(x-\frac{\pi}{2}\right) \;⑤}$$

$$= \frac{-2\times 1}{2(1)^2 - 1} \quad ⑤$$

$$= -2 \qquad \boxed{25}$$

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5. $\dfrac{\mathrm{d}}{\mathrm{d}x}\ln\left(x+\sqrt{x^2+a^2}\right)$
$= \dfrac{1}{\left(x+\sqrt{x^2+a^2}\right)}\cdot\left(1+\dfrac{1}{2}\cdot\dfrac{2x}{\sqrt{x^2+a^2}}\right) \qquad ⑤$

$= \dfrac{1}{\left(x+\sqrt{x^2+a^2}\right)}\cdot\left(\dfrac{x+\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}}\right)$

$= \dfrac{1}{\sqrt{x^2+a^2}} \qquad\qquad ⑤$

Then   I $= \displaystyle\int \dfrac{1}{\sqrt{9x^2+4}}\,\mathrm{d}x = \dfrac{1}{3}\int \dfrac{1}{\sqrt{x^2+\left(\frac{2}{3}\right)^2}}\,\mathrm{d}x \quad ⑤$

$\therefore$ I $= \dfrac{1}{3}\left[\ln\left(x+\sqrt{x^2+\left(\frac{2}{3}\right)^2}\right)\right]+C$; Here C is an arbitrary constant.

$⑤ \quad + \quad ⑤$

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6. $\left(\dfrac{dy}{dx}\right)_T = \dfrac{T(2-3T)}{(1-T)(1-3T)}$

When $T = \dfrac{1}{2}$

$\left(\dfrac{dy}{dx}\right) = \dfrac{\dfrac{1}{2}\left(2-\dfrac{3}{2}\right)}{\left(1-\dfrac{1}{2}\right)\left(1-\dfrac{3}{2}\right)} = -1$ ⑤

Then $(x,y) = \left(\dfrac{1}{8},\dfrac{1}{8}\right)$ ⑤

If any point on the tangent is given by (x,y), the equation of the tangent is

$y - \dfrac{1}{8} = -1\left(x - \dfrac{1}{8}\right)$ ⑤

$4x + 4y - 1 = 0$

Let the tangent and the curve intersect at the point corresponding to $t = T'$.

Then $(x,y) = \left(T'(1-T')^2, T'^2(1-T')\right)$

$\Rightarrow \quad 4T'(1-T')^2 + 4T'^2(1-T') - 1 = 0$ ⑤

$4T'^2 - 4T' + 1 = 0$

$(2T' - 1)^2 = 0$
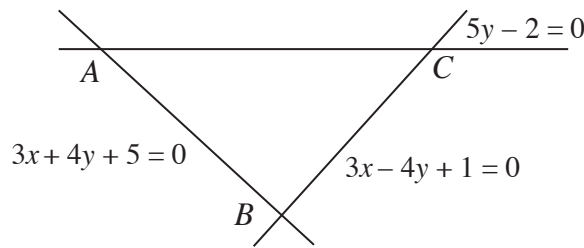
$\Rightarrow T' = \dfrac{1}{2}$; This is the parameter corresponding to the given point. ⑤

Therefore the tangent does not meet the curve again.

That is, the curve is located on one side of the tangent $4x + 4y - 1 = 0$.

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7.

The bisectors of $A\hat{B}C$ are given by $\left|\dfrac{3x+4y+5}{\sqrt{3^2+4^2}}\right| = \left|\dfrac{3x-4y+1}{\sqrt{3^2+(-4)^2}}\right|$. ⑩

i.e., $3x + 4y + 5 = \pm(3x - 4y + 1)$

$8y + 4 = 0 \quad$ and $\quad 6x + 6 = 0$

$2y + 1 = 0 \quad$ and $\quad x + 1 = 0$ ⑤

Since the straight line given by $2y + 1 = 0$ is parallel to the side AC, it is the exterior bisector. ⑤

The straight line $x + 1 = 0$ is perpendicular to the side AC. It is the interior bisector of $A\hat{B}C$. ⑤

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[see page five

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8. For $ax^2 + 2y^2 + bxy + x + 4y + 2c = 0$ to represent a circle,

$a = 2, b = 0$ and $\left(\dfrac{1}{4}\right)^2 + 1^2 - c > 0$. ⑤

$\therefore a = 2, b = 0$ and $c < \dfrac{17}{16}$. ⑤

$\therefore$ The positive integral value of c is 1.

Then the circle is $\quad 2x^2 + 2y^2 + x + 4y + 2 = 0$

That is, $\qquad x^2 + y^2 + \dfrac{1}{2}x + 2y + 1 = 0$

The centre is $\quad \left(-\dfrac{1}{4}, -1\right)$ ⑤

$(x + p)^2 + y^2 = p^2 \quad\Rightarrow\quad x^2 + y^2 + 2px = 0$

The common chord of the two circles is $(\dfrac{1}{2} - 2p)x + 2y + 1 = 0$

Since the given circle is bisected, its centre must be located on the common chord.

⑤

$\therefore \left(\dfrac{1}{2} - 2p\right)\left(-\dfrac{1}{4}\right) + 2(-1) + 1 = 0$

$\dfrac{1}{2}p - \dfrac{1}{8} - 2 + 1 = 0$

$p = \dfrac{9}{4}$ ⑤

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9. $S_1 : x^2 + y^2 - 6x + 8y + 9 = 0 \qquad\Rightarrow\quad C_1 = (3, -4), \quad r_1 = \sqrt{9 + 16 - 9} = 4$ ⎱ ⑤

$S_2 : x^2 + y^2 - r^2 = 0 \qquad\qquad\Rightarrow\quad C_2 = (0, 0), \quad r_2 = r$ ⎰

For the two circles S_1 and S_2 to touch each other $\quad C_1C_2 = r_1 + r_2 \quad$ or $\quad C_1C_2 = |r_1 - r_2|$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 5 = 4 + r \qquad$ or $\qquad 5 = |r - 4|$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad r = 1$ ⑤ $\qquad\qquad r - 4 = \pm 5$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad r = 4 \pm 5$

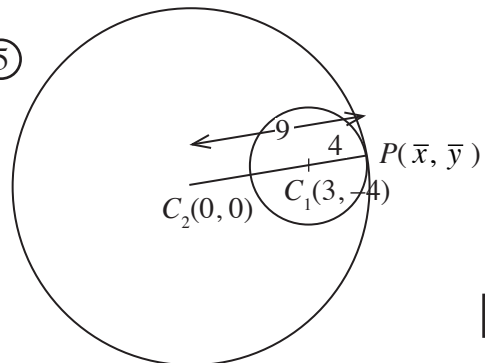$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ Since $r > 0, r = 9$ ⑤

When the circles S_1 and S_2 touch each other internally,
the straight line C_2C_1 is divided externally
at the point of contact P in the ratio $C_2P : PC_1 = 9 : 4$ ⑤

$\therefore P \equiv \left[\dfrac{9 \times 3 - 4 \times 0}{9 - 4}, \dfrac{9 \times (-4) - 4 \times 0}{9 - 4}\right]$

$\equiv \left(\dfrac{27}{5}, -\dfrac{36}{5}\right)$ ⑤

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10. The cosine formula gives,

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$16 = 25 + 36 - 60\cos C \qquad ⑤$$
$$\cos C = \frac{45}{60}$$
$$= \frac{3}{4}$$

The sine formula gives,

$$\frac{6}{\sin B} = \frac{4}{\sin C} \qquad ⑤$$

$$\Rightarrow \sin B = \frac{3}{2}\sin C \qquad\qquad \therefore \ \frac{\sin B}{\sin C} = \frac{3}{2} \, .$$

$$= 2 \times \frac{3}{4} \times \sin C$$

$$= 2\cos C \sin C \qquad ⑤$$

$$= \sin 2C$$

$$\therefore \ \hat{B} = 2\hat{C} \ \text{ or } \ \hat{B} = \pi - 2\hat{C} \quad ; \ (\text{since } 0 < \hat{A}, \ \hat{B}, \ \hat{C} < \pi) \quad ⑤$$
$$\text{Since } \hat{A} \neq \hat{C}, \ \hat{B} \neq \pi - 2\hat{C} \qquad ; \ (\text{since } \hat{A} + \hat{B} + \hat{C} = \pi) \qquad ⑤$$
$$\therefore \ \hat{B} = 2\hat{C}$$

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Part B

11. (a) $\alpha + \beta = -b, \ \alpha\beta = c$ ⑤

$$p + q \ = \ \alpha + \beta + \alpha^2 + \beta^2 \ = \ (\alpha+\beta) + (\alpha+\beta)^2 - 2\alpha\beta \ = \ b^2 - b - 2c \quad ⑤$$

$$pq \ = \ \alpha^3 + \beta^3 + \alpha\beta + (\alpha\beta)^2 \ = \ (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta) + \alpha\beta + (\alpha\beta)^2$$

$$= \ -b^3 + 3cb + c + c^2 \quad ⑤$$

$\therefore$ The quadratic equation whose roots are p and q is,

$$x^2 - (b^2 - b - 2c)x - b^3 + 3cb + c + c^2 \ = \ 0 \qquad ⑤ + ⑤$$

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The discriminant,

$$\Delta_x \ = \ (b^2 - b - 2c)^2 + 4(b^3 - 3bc - c - c^2) \qquad ⑤$$

$$= \ b^4 + b^2 + 4c^2 - 2b^3 + 4bc - 4b^2c + 4b^3 - 12bc - 4c - 4c^2$$

$$= \ b^4 + 2b^3 + b^2 - 4b^2c - 8bc - 4c$$

$$= \ b^2(b^2 + 2b + 1) - 4c(b^2 + 2b + 1)$$

$$= \ (b + 1)^2 (b^2 - 4c) \qquad ⑤$$

When α and β are imaginary, $b^2 - 4c < 0$. ⑤

$\therefore \Delta_x \leq 0$ ⑤

$\therefore \Delta_x = 0$ if and only if $b = -1$. ⑤

That is, p and q are real, if and only if $b = -1$.

Since $\Delta_x = 0, \ p = q$. ⑤

Then $\quad p + q \ = \ (-1)^2 - (-1) - 2c$ ⑤

$$2p \ = \ 2 - 2c$$

$$p \ = \ 1 - c$$

$$\therefore p \ = \ q \ = \ 1 - c$$

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(b) Let $y = \dfrac{(x+2)^2}{x^2 + x + 1}$ ⑤

$$\Rightarrow \qquad (y-1)x^2 + (y-4)x + (y-4) \ = \ 0 \quad ⑤$$

If $y = 1$, then $x = -1$. In this case a quadratic equation does not exist.

$$\Rightarrow \qquad (y-1)\left[\left(x + \frac{y-4}{2(y-1)}\right)^2 + \frac{(y-4)}{(y-1)} - \left(\frac{y-4}{2(y-1)}\right)^2\right] = 0 \qquad (y \neq 1)$$

$$\Rightarrow \qquad (y-1)\left[\left(x + \frac{y-4}{2(y-1)}\right)^2 + \frac{4(y-1)(y-4) - (y-4)^2}{4(y-1)^2}\right] = 0$$

$$\Rightarrow \qquad (y-1)\left[\left(x + \frac{y-4}{2(y-1)}\right)^2 + \frac{(y-4)(4y-4-y+4)}{4(y-1)^2}\right] = 0$$

$$\Rightarrow \qquad (y-1)\left[\left(x + \frac{y-4}{2(y-1)}\right)^2 + \frac{3y(y-4)}{4(y-1)^2}\right] = 0 \qquad ⑤$$

[see page eight

For all real x, $\dfrac{3y(y-4)}{4(y-1)^2} \le 0$

$\Rightarrow\quad 3y(y-4) \le 0$, ⑤ since $(y-1)^2 > 0$

$\Rightarrow\quad 0 \le y \le 4$ ⑤

$\Rightarrow\quad 0 \le \dfrac{(x+2)^2}{x^2+x+1} \le 4$

$\Rightarrow\quad y_{min} = 0$ and $y_{max} = 4$ ⑤

The graph of $y = \dfrac{(x+2)^2}{x^2+x+1}$ is continuous, since $x^2 + x + 1 = \left(x + \dfrac{1}{2}\right)^2 + \dfrac{3}{4} > 0$.
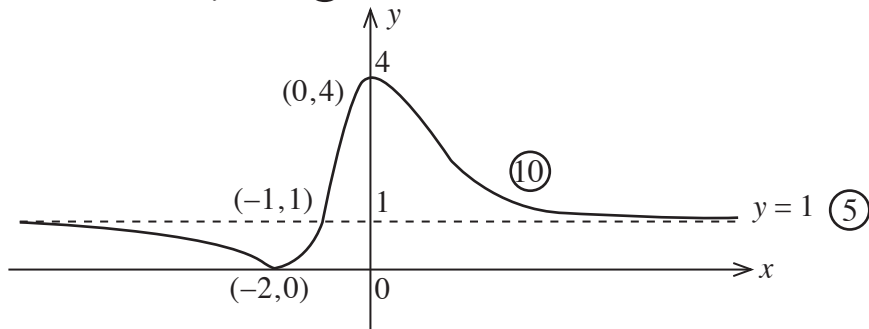
$(0, 4)$ is a maximum. ⑤

$(-2, 0)$ is a minimum. ⑤

When $y = 1$, $x^2 + x + 1 = x^2 + 4x + 4$ $\Rightarrow$ $3x + 3 = 0$ $\Rightarrow$ $x = -1$

$y = \dfrac{x^2 + 4x + 4}{x^2 + x + 1} = \dfrac{1 + \frac{4}{x} + \frac{4}{x^2}}{1 + \frac{1}{x} + \frac{1}{x^2}}$, for $x \ne 0$

When $x \to \pm\infty$, $y \to 1$ ⑤

⑩ 　 $y = 1$ ⑤

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(c)　If $x^2 + kx + 1$ is a factor of $x^4 - 12x^2 + 8x + 3$, there is $\lambda \in \mathbb{Z}$ such that,

$x^4 - 12x^2 + 8x + 3 \equiv (x^2 + kx + 1)(x^2 + \lambda x + 3)$

⑤

$\Rightarrow\quad k + \lambda = 0$ and $\lambda + 3k = 8$　⑤

$\Rightarrow\quad k = 4$　⑤

$\Rightarrow\quad \lambda = -4$

$x^4 - 12x^2 + 8x + 3 = 0$

$(x^2 + 4x + 1)(x^2 - 4x + 3) = 0$　⑤

$x^2 + 4x + 1 = 0$　or　$x^2 - 4x + 3 = 0$

$x = \dfrac{-4 \pm \sqrt{16-4}}{2}$　or　$(x-1)(x-3) = 0$

$= -2 \pm \sqrt{3}$ ⑤　or　$x = 1$ or $x = 3$ ⑤

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[see page nine

12. (a) Since each child should get at least three rupees, an amount of 15 rupees must be reserved compulsorily. Then the remaining three rupees can be divided among the five children in different ways as given below. ⑤

	Division of money					No. of ways	
(i)	3	0	0	0	0 ⑤	$\dfrac{5!}{4!} = 5$	⑤
(ii)	2	1	0	0	0 ⑤	$\dfrac{5!}{3!} = 20$	⑤
(iii)	1	1	1	0	0 ⑤	$\dfrac{5!}{3!2!} = 10$	⑤

∴ Total number of ways $= 5 + 20 + 10$ ⑤
$\qquad\qquad\qquad\qquad\; = 35$

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(b)
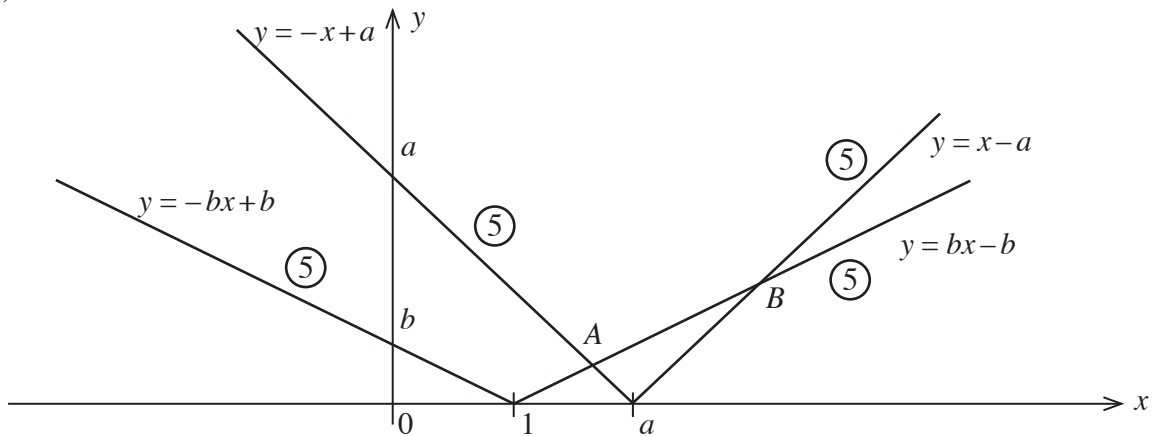
$a > b > 0$
Since the solution set of $b|x-1| > |x-a|$ is $\{x\,|\,3 < x < 7\;;\;x \in \mathbb{R}\}$,
$A \equiv (3, y_A)\;;\;B \equiv (7, y_B).$ ⑤
By considering y_A, $-3 + a = 3b - b \Rightarrow a - 2b = 3$ ——— (1) ⑤
Similarly by considering y_B, $7 - a = 7b - b \Rightarrow a + 6b = 7$ ——— (2) ⑤
From (1) and (2), $a = 4$, $b = \dfrac{1}{2}$ ⑤

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(c) Let $u_r \equiv \dfrac{A}{r+1} + \dfrac{B}{r+2} + \dfrac{C}{r+3} \equiv \dfrac{3r+1}{(r+1)(r+2)(r+3)}.$ ⑤

Then $A(r+2)(r+3) + B(r+1)(r+3) + C(r+1)(r+2) \equiv 3r + 1$
By considering the coefficients of,
r^2 ; $\qquad A + B + C = 0$
r ; $\qquad\; 5A + 4B + 3C = 3$ ⑩
Constant $\quad 6A + 3B + 2C = 1$

By solving the above equations ; $A = -1, B = 5, C = -4$ ⑩

$\therefore u_r \equiv -\dfrac{1}{r+1} + \dfrac{5}{r+2} - \dfrac{4}{r+3}$

[see page ten

$$\equiv -\left(\frac{1}{r+1} - \frac{1}{r+2}\right) + 4\left(\frac{1}{r+2} - \frac{1}{r+3}\right) \quad \textcircled{5}$$

This is in the form of $\lambda[f(r) - f(r+1)] + \mu[f(r+1) - f(r+2)]$.

Here $\lambda = -1$, $\mu = 4$ and $f(r) = \left(\frac{1}{r+1}\right)$. $\textcircled{5}$

$\textcircled{5}$

$$\begin{aligned}
u_r &\equiv \lambda[f(r) - f(r+1)] + \mu[f(r+1) - f(r+2)] \\
\therefore u_1 &= \lambda[f(1) - f(2)] + \mu[f(2) - f(3)] \\
u_2 &= \lambda[f(2) - f(3)] + \mu[f(3) - f(4)] \quad \textcircled{5} \\
&\vdots \\
u_{n-1} &= \lambda[f(n-1) - f(n)] + \mu[f(n) - f(n+1)] \\
u_n &= \lambda[f(n) - f(n+1)] + \mu[f(n+1) - f(n+2)] \quad \textcircled{5}
\end{aligned}$$

By adding,

$$\begin{aligned}
\sum_{r=1}^{n} u_r &= \lambda[f(1) - f(n+1)] + \mu[f(2) - f(n+2)] \\
&= -1\left[\frac{1}{2} - \frac{1}{n+2}\right] + 4\left[\frac{1}{3} - \frac{1}{n+3}\right] \quad \textcircled{5} \\
&= \frac{5}{6} + \frac{1}{n+2} - \frac{4}{n+3} \\
&= \frac{5}{6} - \frac{3n+5}{(n+2)(n+3)}
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty} \sum_{r=1}^{n} u_r &= \lim_{n\to\infty}\left[\frac{5}{6} + \frac{1}{n+2} - \frac{4}{n+3}\right] \\
&= \frac{5}{6} \quad \textcircled{5}
\end{aligned}$$

$\Rightarrow$ Since $\displaystyle\lim_{n\to\infty}\sum_{r=1}^{n} u_r$ is finite, the series is convergent. $\textcircled{5}$

Further more, for all $r \in \mathbf{Z}^+$, $u_r > 0$.

$$\therefore u_1 \le S_n < S_\infty \quad \textcircled{5}$$

$$\Rightarrow \frac{1}{6} \le \frac{5}{6} - \frac{3n+5}{(n+2)(n+3)} < \frac{5}{6}$$

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13. (a) $\mathbf{P} = \begin{pmatrix} -5 & 3 \\ 6 & -2 \end{pmatrix}$

$\mathbf{P} - \lambda\mathbf{I} = \begin{pmatrix} -5 & 3 \\ 6 & -2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} -5-\lambda & 3 \\ 6 & -2-\lambda \end{pmatrix}$ ⑤

$\det(\mathbf{P} - \lambda\mathbf{I}) = 0$

$\Rightarrow \begin{vmatrix} -5-\lambda & 3 \\ 6 & -2-\lambda \end{vmatrix} = 0$ ⑤

$\Rightarrow (5+\lambda)(2+\lambda) - 18 = 0$ ⑤

$\Rightarrow \lambda^2 + 7\lambda - 8 = 0$

$\Rightarrow (\lambda - 1)(\lambda + 8) = 0$

$\Rightarrow \lambda = 1 \quad \text{or} \quad \lambda = -8$ ⑤

$\mathbf{PX} = \lambda\mathbf{X}$

$\Rightarrow \begin{pmatrix} -5 & 3 \\ 6 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} -5x+3y \\ 6x-2y \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix}$

When $\lambda = 1$, $\left.\begin{array}{r} -5x + 3y = x \\ 6x - 2y = y \end{array}\right\} \Rightarrow \begin{array}{r} -6x + 3y = 0 \\ 6x - 3y = 0 \end{array}$ ⑤

Since the two equations above are equivalent,
when $x = t$, $y = 2t$; Here t is a real parameter.

$\therefore X = \begin{pmatrix} t \\ 2t \end{pmatrix} = t\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ ⑤

When $\lambda = -8$, $\left.\begin{array}{r} -5x + 3y = -8x \\ 6x - 2y = -8y \end{array}\right\} \Rightarrow \begin{array}{r} 3x + 3y = 0 \\ 6x + 6y = 0 \end{array}$ ⑤

Since the above two equations are also equivalent,
when $x = T$ then $y = -T$; Here T is a real parameter.

$\therefore X = \begin{pmatrix} T \\ -T \end{pmatrix} = T\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ⑤

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....

(b)
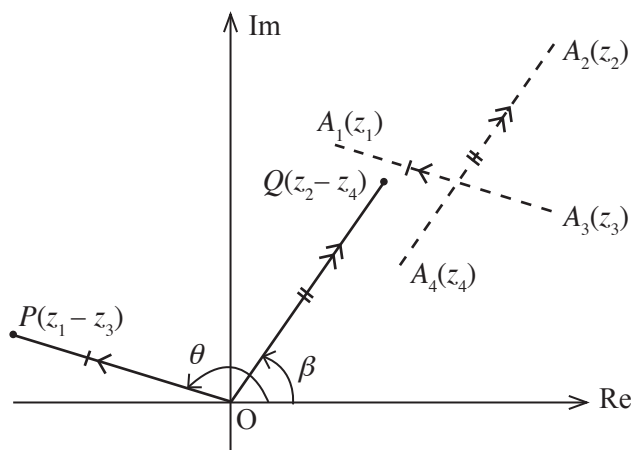
P and Q are two points such that $OP \parallel A_3 A_1$, $OP = A_3 A_1$ and $OQ \parallel A_4 A_2$, $OQ = A_4 A_2$ respectively.

Then $z_P = z_1 - z_3$ and $z_Q = z_2 - z_4$. ⑩

$$\left| \frac{z_1 - z_3}{z_2 - z_4} \right| = \frac{|z_1 - z_3|}{|z_2 - z_4|} = \frac{OP}{OQ} = \frac{A_1 A_3}{A_2 A_4}$$ ⑩

$$\arg\left(\frac{z_1 - z_3}{z_2 - z_4} \right) = \arg(z_1 - z_3) - \arg(z_2 - z_4) = \theta - \beta = P\hat{O}Q$$

When $\beta > \theta$, $|\theta - \beta| = P\hat{O}Q$.

$P\hat{O}Q$ = the angle between $A_1 A_3$ and $A_2 A_4$. ⑩

Therefore, for $\dfrac{z_1 - z_3}{z_2 - z_4}$ to be purely imaginary, $A_1 A_3 \perp A_2 A_4$. ⑤

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$z^2 - 2z + 2 = 0 \quad \Rightarrow \quad z = 1 \pm i$

$z^2 - 2az + b = 0 \quad \Rightarrow \quad z = a \pm i\sqrt{b - a^2}$

(i) If $C\hat{O}D = \dfrac{\pi}{2}$,

$\dfrac{a + i\sqrt{b - a^2}}{a - i\sqrt{b - a^2}}$ is purely imaginary. ⑤

$$\frac{a + i\sqrt{b - a^2}}{a - i\sqrt{b - a^2}} = \frac{a^2 - (b - a^2) + 2ia\sqrt{b - a^2}}{a^2 + (b - a^2)}$$

For this number to be purely imaginary, the real part $= 0 \Rightarrow 2a^2 = b$ ⑩
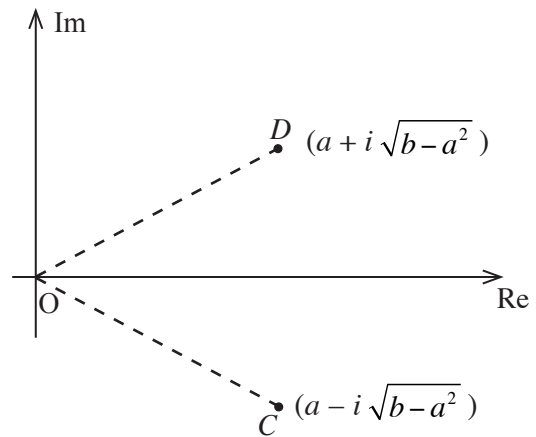
(ii) $OA = OB = OC = OD$

$\therefore \ z_A \bar{z}_A = z_D \bar{z}_D$ ⑤

$\Rightarrow \quad 2 = a^2 + b - a^2$ ⑩

$\Rightarrow \quad b = 2$

(iii) When $ABCD$ is a square and its centre is O,
by symmetry, ⑤
$a = -1$ and $\sqrt{b - a^2} = 1$
$\Rightarrow \quad b = 2$ ⑩

$\boxed{45}$

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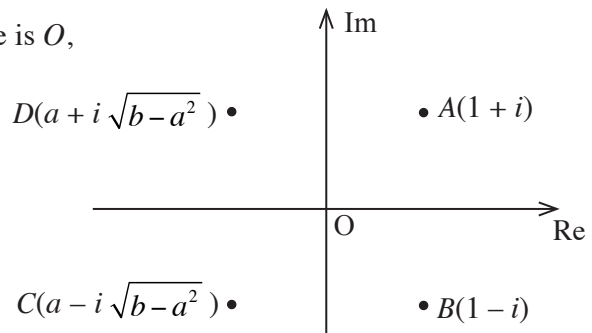
(c) $\arg[(z+i)i] = \dfrac{2\pi}{3}$

$\arg(z+i) + \arg i = \dfrac{2\pi}{3}$

$\therefore \arg(z+i) = \dfrac{2\pi}{3} - \dfrac{\pi}{2} = \dfrac{\pi}{6}$ ⑤

$\Rightarrow \arg(z-(-i)) = \dfrac{\pi}{6}$ ⑤

$\therefore P\hat{A}X' = \dfrac{\pi}{6}; P \neq A$ ⑤
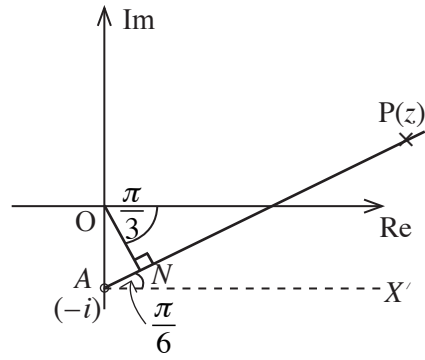
The locus of P is a line segment as shown in the figure.

$OP = |z|$ ⑤

$\therefore |z|_{\text{least}} = ON = 1.\cos\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}$ ⑤

$z_N = \left\{ \dfrac{\sqrt{3}}{2}\cos\dfrac{\pi}{3} - i\dfrac{\sqrt{3}}{2}\sin\dfrac{\pi}{3} \right\}$

$\quad = \dfrac{\sqrt{3}}{4} - \dfrac{3}{4}i$ ⑤   ☐30

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14. (a) $f(x) = \dfrac{3-4x}{x^2+1}$, $x \in \mathbb{R}$

$f'(x) = \dfrac{(x^2+1)(-4) - (3-4x)2x}{(x^2+1)^2}$ ⑩ $= \dfrac{2(2x+1)(x-2)}{(x^2+1)^2}$

Since at $x = -\dfrac{1}{2}$ and $x = 2$, $f'(x) = 0$, ⑩ there exist two turning points.

For all $x \in \mathbb{R}$, the function f is continuous.

x	$-\infty < x < -\dfrac{1}{2}$	$-\dfrac{1}{2} < x < 2$	$2 < x < \infty$
$f'(x)$	$\dfrac{(-)(-)}{(+)} > 0$	$\dfrac{(+)(-)}{(+)} < 0$	$\dfrac{(+)(+)}{(+)} > 0$	⑩

$x = -\dfrac{1}{2}$, $y = 4$ is a maximum. ⑤

$x = 2, y = -1$ is a minimum. ⑤

When $x = 0$, $y = 3$

When $x = \dfrac{3}{4}$, $y = 0$ ⑤

$y = \dfrac{3/x^2 - 4/x}{1 + 1/x^2}$

$x \to \pm\infty$, $y \to 0$ ⑤

$y = f(x) = \dfrac{3-4x}{x^2+1}$ ⑩

$|3-4x|e^x - x^2 - 1 = 0$ ———— (1)
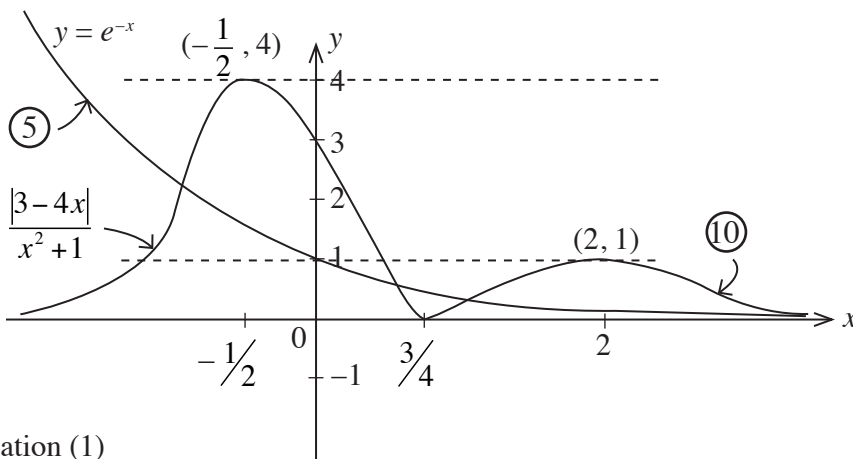
- **14** -
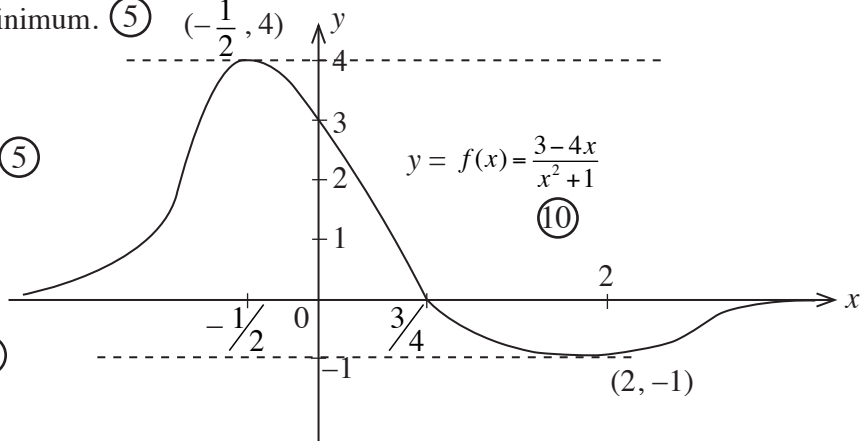
$$e^{-x} = \frac{|3-4x|}{x^2+1} \quad ⑤$$

The roots of the equation (1) are given by the x - coordinates of the intersection points of the curves $y = e^{-x}$ and
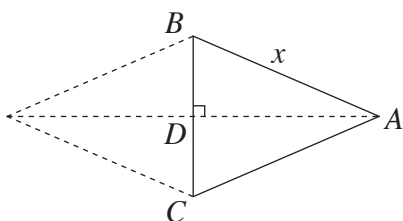
$$y = \frac{|3-4x|}{x^2+1}. \quad ⑤$$

According to the graph, equation (1) has at least three distinct real roots. ⑤

90

(b)

Let the volume of the generated solid be V, when $AB = x$.

Then $BC = 2s - 2x$,

$$BD = \frac{2s-2x}{2} = s - x \quad ⑤$$

$$AD = \sqrt{AB^2 - BD^2} = \sqrt{x^2 - (s-x)^2} = \sqrt{2sx - s^2}$$

$$= \sqrt{s(2x-s)} \quad ⑤ \;;\; x > \frac{s}{2} \quad ⑤$$

$$V = \frac{1}{3}\pi AD^2 \cdot 2BD \quad ⑤$$

$$= \frac{1}{3}\pi s(2x-s) \cdot 2(s-x) \quad ⑤$$

$$= \frac{2\pi}{3} s(2x-s)(s-x)$$

$$\frac{dV}{dx} = \frac{2}{3}\pi s[(2x-s)(-1) + (s-x)2] \quad ⑤$$

$$= \frac{2}{3}\pi s[-2x + s + 2s - 2x]$$

$$= \frac{2}{3}\pi s(3s - 4x) \quad ⑤$$

When $x = \frac{3}{4}s$, $\frac{dV}{dx} = 0$. ⑤

When $\frac{s}{2} < x < \frac{3s}{4}$, $\frac{dV}{dx} > 0$; $\therefore V$ increases.

When $\frac{3s}{4} < x < s$, $\frac{dV}{dx} < 0$; $\therefore V$ decreases. ⑩

$\therefore$ When $x = \frac{3}{4}s$, V is a maximum. ⑤

$\therefore$ The volume is maximum, when AB is of length $\frac{3}{4}s$. ⑤

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[see page fifteen

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15. (a) $\dfrac{x^2+3x+5}{(x-1)(x+2)} \equiv A + \dfrac{B}{x-1} + \dfrac{C}{x+2}$

$\Rightarrow x^2 + 3x + 5 \equiv A(x-1)(x+2) + B(x+2) + C(x-1)$

By comparing the coefficents of x^2, $A = 1$ ⑤

When $x = 1$, $B = 3$ ⑤

When $x = -2, C = -1$ ⑤

$\therefore \displaystyle\int_0^2 \dfrac{x^2+3x+5}{(x-1)(x+2)}\,\mathrm{d}x = \displaystyle\int_0^2 \mathrm{d}x + 3\displaystyle\int_0^2 \dfrac{1}{x-1}\,\mathrm{d}x - \displaystyle\int_0^2 \dfrac{1}{x+2}\,\mathrm{d}x$ ⑤

$= \left[x\right]_0^2 + 3\left[\ln|x-1|\right]_0^2 - \left[\ln|x+2|\right]_0^2$ ⑩

$= 2 + 3\,[\ln 1 - \ln 1] - [\ln 4 - \ln 2]$

$= 2 - \ln 2$ ⑤

$\boxed{35}$

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(b) $\displaystyle\int e^{2x}\sin 3x\,\mathrm{d}x = \displaystyle\int \sin 3x\,\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{e^{2x}}{2}\right)\mathrm{d}x$ ⑤

$= \dfrac{1}{2}e^{2x}\sin 3x - \displaystyle\int \dfrac{1}{2}e^{2x}3\cos 3x\,\mathrm{d}x + C$ ⑩

$= \dfrac{1}{2}e^{2x}\sin 3x - \dfrac{3}{2}\left[\dfrac{1}{2}e^{2x}\cos 3x - \dfrac{1}{2}\displaystyle\int e^{2x}(-3\sin 3x)\mathrm{d}x\right] + C'$ ⑩

$= \dfrac{1}{2}e^{2x}\sin 3x - \dfrac{3}{4}e^{2x}\cos 3x - \dfrac{9}{4}\displaystyle\int e^{2x}\sin 3x\,\mathrm{d}x + C'$ ⑤

$\Rightarrow \dfrac{13}{4}\displaystyle\int e^{2x}\sin 3x\,\mathrm{d}x = \dfrac{1}{4}e^{2x}(2\sin 3x - 3\cos 3x) + C'$ ⑩

$\Rightarrow \displaystyle\int e^{2x}\sin 3x\,\mathrm{d}x = \dfrac{1}{13}e^{2x}(2\sin 3x - 3\cos 3x) + C$; Here C is an arbitrary constant.

⑤

$\boxed{45}$

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(c) $\mathrm{I} = \displaystyle\int_0^1 \dfrac{1}{x+\sqrt{1-x^2}}\,\mathrm{d}x$

$x = \sin\theta$

$\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = \cos\theta$

When $x = 1$, $\theta = \dfrac{\pi}{2}$

When $x = 0$, $\theta = 0$ ⑤

$\therefore \mathrm{I} = \displaystyle\int_0^{\frac{\pi}{2}} \dfrac{\cos\theta}{\sin\theta + \sqrt{1-\sin^2\theta}}\,\mathrm{d}\theta$ ⑤

$= \displaystyle\int_0^{\frac{\pi}{2}} \dfrac{\cos\theta}{\sin\theta + \cos\theta}\,\mathrm{d}\theta$ ———— (1)

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When $\theta = \dfrac{\pi}{2} - t$,

$\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = -1$ ⑤

When $\theta = 0$, $t = \dfrac{\pi}{2}$

When $\theta = \dfrac{\pi}{2}$, $t = 0$ ⑤

$\therefore$ I $= \displaystyle\int_{\frac{\pi}{2}}^{0} \dfrac{\sin t}{\cos t + \sin t}\,(-\mathrm{d}t)$

$= \displaystyle\int_{0}^{\frac{\pi}{2}} \dfrac{\sin t}{\cos t + \sin t}\,\mathrm{d}t$ ⑤

Replacing t by θ, I $= \displaystyle\int_{0}^{\frac{\pi}{2}} \dfrac{\sin\theta}{\cos\theta + \sin\theta}\,\mathrm{d}\theta$ —————— (2) ⑤

$(1) + (2)$; $\quad 2\text{I} = \displaystyle\int_{0}^{\frac{\pi}{2}}\mathrm{d}\theta = \left[\theta\right]_{0}^{\pi/2}$ ⑤ + ⑤

$\therefore$ I $= \dfrac{\pi}{4}$

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(d) Let $S = \{(x, y) : x^2 \le y \le |x|\}$.
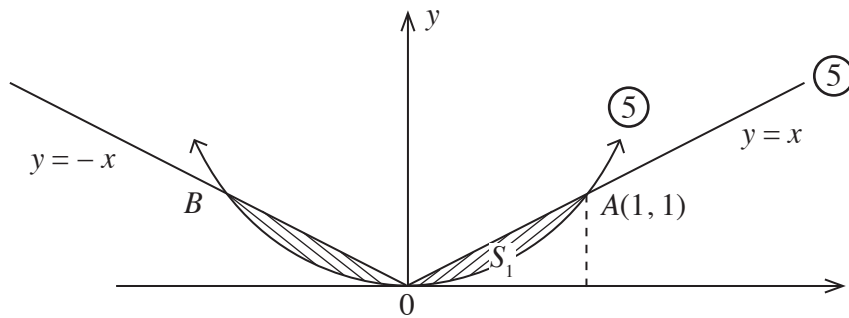
By solving $y = x$ and $y = x^2$,

$A \equiv (1, 1)$ ⑤

$S_1 = \dfrac{1}{2} \times 1 \times 1 - \displaystyle\int_{0}^{1} x^2\,\mathrm{d}x$ ⑤

$= \dfrac{1}{2} - \left[\dfrac{x^3}{3}\right]_{0}^{1} = \dfrac{1}{2} - \left(\dfrac{1}{3}\right)$ ⑤

$= \dfrac{3-2}{6} = \dfrac{1}{6}$

$\therefore$ The enclosed area $= 2S_1 = \dfrac{1}{3}$ square units ⑤

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16. (a) $A \equiv \left(\dfrac{4}{3}, \dfrac{5}{3}\right)$ ⑤

   $B \equiv (4, 3)$ ⑤

   The equation of the line AD,

   $y - \dfrac{5}{3} = -1\left(x - \dfrac{4}{3}\right)$ ⑩

   $y - \dfrac{5}{3} = -x + \dfrac{4}{3}$

   $x + y - 3 = 0$ ⑤
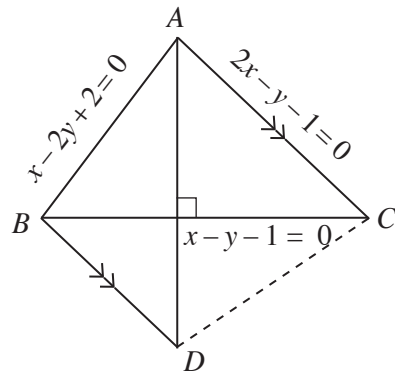
   Since $BD /\!/ AC$, the equation of the line BD can be taken as $2x - y - k = 0$. Here k is a constant. ⑤

   Since this passes through B, $8 - 3 - k = 0$

   $\therefore k = 5$ ⑤

   $\therefore$ the equation of the line BD is $2x - y - 5 = 0$ ⑤

   $C \equiv (0, -1)$ ⑤

   $D \equiv \left(\dfrac{8}{3}, \dfrac{1}{3}\right)$ ⑤

   $\therefore$ the gradient of $CD = \dfrac{\frac{1}{3}+1}{\frac{8}{3}-0} = \dfrac{1}{2}$ ⑤

   The gradient of $AB = \dfrac{3-\frac{5}{3}}{4-\frac{4}{3}} = \dfrac{1}{2}$ ⑤

   $\therefore AB /\!/ CD$ ⑤

   Also, since $BD /\!/ AC$, $ABCD$ is a parallelogram. ⑤

   Since its diagonals are perpendicular, $ABDC$ is a rhombus. ⑤

   $\boxed{75}$

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(b) $|r_1 - r_2| < C_1 C_2 < r_1 + r_2$ ⑩

   If the circles $x^2 + y^2 + 6x + 2fy = 0$ and $x^2 + y^2 - 2y - 3 = 0$ intersect each other orthogonally,

   $2.3.0 + 2f(-1) = 0 + (-3)$ ⑩

   $\Rightarrow f = \dfrac{3}{2}$

   Let the circles $S_1 \equiv x^2 + y^2 + 6x + 2fy = 0$ and $S_2 \equiv x^2 + y^2 - 2y - 3 = 0$ intersect each other at the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

   $S_1 + \lambda S_2 = 0 \Rightarrow (1+\lambda)x^2 + (1+\lambda)y^2 + 6x + 2(f-\lambda)y - 3\lambda = 0$

   $\therefore$ When λ is a parameter, this equation represents a circle. ⑤

   Since the circles $S_1 = 0$ and $S_2 = 0$ pass through the point $P_1(x_1, y_1)$,

   $x_1^2 + y_1^2 + 6x_1 + 2fy_1 = 0$ ——— (1)

   $x_1^2 + y_1^2 - 2y_1 - 3 = 0$ ——— (2)

   From (1) + λ(2), $(1+\lambda)x_1^2 + (1+\lambda)y_1^2 + 6x_1 + 2(f-\lambda)y_1 - 3\lambda = 0$

   That is, the circle $S_1 + \lambda S_2 = 0$ passes through the point $P_1(x_1, y_1)$. ⑩

It can be shown similarly that the circle $S_1 + \lambda S_2 = 0$ passes through the point $P_2(x_2, y_2)$. ⑤

$\therefore S_1 + \lambda S_2 = 0$ represents any circle that passes through the intersection points of the two circles $S_1 = 0$ and $S_2 = 0$. Here λ is a parameter.

(i) $S_1 + \lambda S_2 \equiv (1+\lambda)x^2 + (1+\lambda)y^2 + 6x + 2(f - \lambda)y - 3\lambda = 0$. Here $f = \dfrac{3}{2}$

When this passes through $(-2, 2)$, $(1+\lambda)4 + (1+\lambda)4 - 12 + 2(\dfrac{3}{2} - \lambda)(2) - 3\lambda = 0$ ⑩
$\Rightarrow 8 - 12 + 6 + 8\lambda - 4\lambda - 3\lambda = 0$
$\Rightarrow 2 + \lambda = 0$
$\Rightarrow \lambda = -2$

$\therefore$ The equation of the required circle is $-x^2 - y^2 + 6x + 7y + 6 = 0$
$$x^2 + y^2 - 6x - 7y - 6 = 0 \quad ⑤$$

(ii) The equation of the common chord is given by $S_1 - S_2 = 0$
$$6x + (2f + 2)y + 3 = 0$$
$$6x + 5y + 3 = 0 \quad ⑤$$
$S_1 + \lambda S_2 = (1+\lambda)x^2 + (1+\lambda)y^2 + 6x + 2(f - \lambda)y - 3\lambda = 0$
Centre $\equiv \left[-\dfrac{3}{1+\lambda}, \dfrac{\lambda - f}{1+\lambda}\right]$ ⑤

Since the centre of the smallest circle lies on the common chord,

$6\left(-\dfrac{3}{1+\lambda}\right) + 5\left(\dfrac{\lambda - f}{1+\lambda}\right) + 3 = 0$. Here $f = \dfrac{3}{2}$ ⑤

$-18 + 5\lambda - 5f + 3 + 3\lambda = 0$
$8\lambda = 18 - 3 + \dfrac{15}{2} = \dfrac{45}{2} \Rightarrow \lambda = \dfrac{45}{16}$

$\therefore$ The equation of the required circle is $\dfrac{61}{16}x^2 + \dfrac{61}{16}y^2 + 6x + 2\left(\dfrac{3}{2} - \dfrac{45}{16}\right)y - \dfrac{135}{16} = 0$

$$61x^2 + 61y^2 + 96x - 42y - 135 = 0 \quad ⑤$$

$\boxed{75}$

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17. (a) $\cos A + \cos B + \cos C = \dfrac{3}{2}$

$1 - 2\sin^2 \dfrac{A}{2} + 2\cos\dfrac{B+C}{2}\cos\dfrac{B-C}{2} = \dfrac{3}{2}$ ⑩

Since $A + B + C = \pi$, $2\cos\left(\dfrac{\pi}{2} - \dfrac{A}{2}\right)\cos\dfrac{B-C}{2} - 2\sin^2\dfrac{A}{2} = \dfrac{1}{2}$

$2\sin\dfrac{A}{2}\left[\cos\dfrac{B-C}{2} - \sin\dfrac{A}{2}\right] = \dfrac{1}{2}$ ⑤

$\cos\dfrac{B-C}{2} - \sin\dfrac{A}{2} = \dfrac{1}{4\sin\dfrac{A}{2}}$

$\Rightarrow \cos\dfrac{B-C}{2} = \sin\dfrac{A}{2} + \dfrac{1}{4\sin\dfrac{A}{2}}$

[see page nineteen

$$= \frac{4\sin^2\frac{A}{2}+1}{4\sin\frac{A}{2}} \qquad ⑤$$

$$= \frac{\left(1-2\sin\frac{A}{2}\right)^2 + 4\sin\frac{A}{2}}{4\sin\frac{A}{2}} \qquad ⑤$$

$$= \frac{\left(1-2\sin\frac{A}{2}\right)^2}{4\sin\frac{A}{2}} + 1$$

Since $\cos\frac{B-C}{2} \le 1,$ $\qquad \dfrac{\left(1-2\sin\frac{A}{2}\right)^2}{4\sin\frac{A}{2}} + 1 \le 1 \qquad ⑤$

$$\Rightarrow \frac{\left(1-2\sin\frac{A}{2}\right)^2}{\sin\frac{A}{2}} \le 0 \qquad ⑤$$

Since $0<A<\pi,$ $\sin\frac{A}{2}>0$. $\therefore$ $1-2\sin\frac{A}{2}=0$.

Then, $\cos\frac{B-C}{2} = 1$ and $\sin\frac{A}{2} = \frac{1}{2}$ $\qquad ⑤$

$\Rightarrow B = C$ and $A = \dfrac{\pi}{3}$; since $0 < A < \pi,$
$\qquad\qquad ⑤$

$\Rightarrow A = B = C = \dfrac{\pi}{3}$

$\Rightarrow$ Triangle ABC is equilateral.

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(b) $\quad f(\theta) \quad \equiv \quad 3\cos^2\theta + 10\sin\theta\cos\theta + 27\sin^2\theta$

$\qquad\qquad \equiv \quad \dfrac{3}{2}[1 + \cos2\theta] + 5\sin2\theta + \dfrac{27}{2}[1 - \cos2\theta] \quad ⑤$

$\qquad\qquad \equiv \quad 15 - 12\cos2\theta + 5\sin2\theta$

$\qquad\qquad \equiv \quad 15 - 13\left[\dfrac{12}{13}\cos2\theta - \dfrac{5}{13}\sin2\theta\right] \qquad ⑤$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad ⑤$

$\qquad\qquad \equiv \quad 15 - 13[\cos2\theta\cos\alpha - \sin2\theta\sin\alpha]$; here $0 < \alpha < \dfrac{\pi}{2}$ such that
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cos\alpha = \dfrac{12}{13}$ and $\sin\alpha = \dfrac{5}{13}$ $\quad ⑤$

$\qquad\qquad \equiv \quad 15 - 13\cos(2\theta + \alpha)$

$\therefore f(\theta)$ is of the form $a + b\cos(2\theta + \alpha).$

Here $a = 15, b = -13, \alpha = \cos^{-1}\left(\dfrac{12}{13}\right) = \sin^{-1}\left(\dfrac{5}{13}\right).$ $0 < \alpha < \dfrac{\pi}{2}$
$\qquad\qquad\quad ⑤ \qquad\qquad ⑤$

$f(\theta) \equiv 15 - 13\cos(2\theta + \alpha)$; here $\alpha = \cos^{-1}\left(\dfrac{12}{13}\right) = \sin^{-1}\left(\dfrac{5}{13}\right)$

$f(\theta)$ is continuous for all $\theta.$

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When $\theta = 0$, $f(0) = 15 - 13\cos\alpha = 15 - 12 = 3$

When $\theta = \pi$, $f(\pi) = 15 - 13\cos(2\pi + \alpha) = 15 - 13\cos\alpha = 3$

$-1 \le \cos(2\theta + \alpha) \le 1 \quad \Rightarrow \quad -13 \le 13\cos(2\theta + \alpha) \le 13 \quad \Rightarrow \quad -13 \le -13\cos(2\theta + \alpha) \le 13$

⑤ $\quad \Rightarrow \quad 2 \le 15 - 13\cos(2\theta + \alpha) \le 28$

$\Rightarrow \quad 2 \le f(\theta) \le 28$ ⑤

At the minimum point, $f(\theta) = 2$. Then $\cos(2\theta + \alpha) = 1 \Rightarrow 2\theta + \alpha = 2n\pi$; here $n \in \mathbb{Z}$ ⑤

$$\theta = n\pi - \frac{\alpha}{2} = \pi - \frac{\alpha}{2}, \text{ since } \theta \in [0, \pi]$$ ⑤

At the maximum point, $f(\theta) = 28$. Then $\cos(2\theta + \alpha) = -1 \Rightarrow 2\theta + \alpha = 2n\pi \pm \pi$, $n \in \mathbb{Z}$ ⑤

$$\theta = n\pi \pm \frac{\pi}{2} - \frac{\alpha}{2} = \frac{\pi}{2} - \frac{\alpha}{2}, \text{ since } \theta \in [0, \pi]$$ ⑤

$f(\theta) \ne 0$
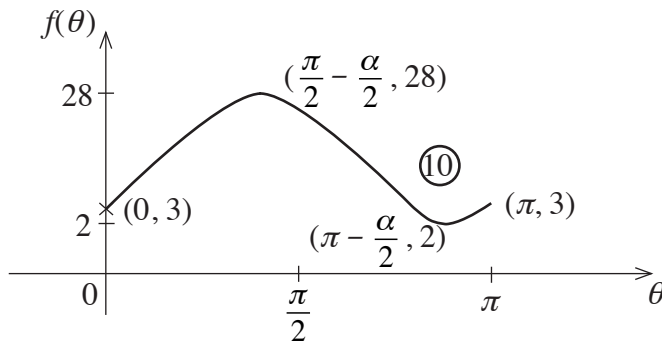
The equation $f(\theta) - k = 0$

(i) has exactly one solution when, $k = 2$ and $k = 28$. ⑤

(ii) has two solutions when, $2 < k < 3$ and $3 < k < 28$. ⑤

(iii) has three solutions when $k = 3$. ⑤

(iv) has no solutions when, $k < 2$ and $k > 28$. ⑤

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(c) $\sin^{-1}\sqrt{\dfrac{2}{3}} - \sin^{-1}x = \dfrac{\pi}{2}$

⑤

Let $\alpha = \sin^{-1}\sqrt{\dfrac{2}{3}}$ and $\beta = \sin^{-1}x$.

Then $\alpha - \beta = \dfrac{\pi}{2} \Rightarrow \alpha = \dfrac{\pi}{2} + \beta$

$\Rightarrow \cos\alpha = \cos\left(\dfrac{\pi}{2} + \beta\right)$

$\qquad = -\sin\beta$ ⑤

$\Rightarrow \dfrac{1}{\sqrt{3}} = -x$

$\Rightarrow x = -\dfrac{1}{\sqrt{3}}$ ⑤

$\alpha = \sin^{-1}\sqrt{\dfrac{2}{3}} \qquad 0 < \alpha < \dfrac{\pi}{2}$

$\sin\alpha = \sqrt{\dfrac{2}{3}}$

$\cos\alpha = \sqrt{1 - \dfrac{2}{3}}$

$\qquad = \sqrt{\dfrac{1}{3}}$

15

Part A

1. The path of the boat relative to the ship:

$DC = \sqrt{24^2 + 7^2}$ km $= 25$ km

$\tan\alpha = \dfrac{7}{24}$ ⑤

S - ship, B - boat, E - earth
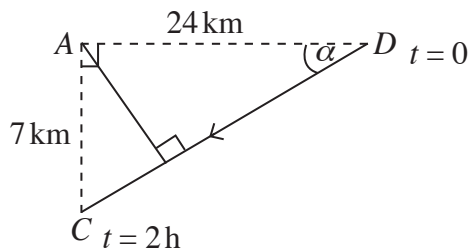
(i) $V_{B,S} = $ ⟋ $\dfrac{25}{2}$ km h^{-1} ⑤

The shortest distance between the ship and the boat $= 24\sin\alpha$ km

$= 24 \cdot \dfrac{7}{25}$ km

$= \dfrac{168}{25}$ km ⑤

(ii) $V_{B,E} = V_{B,S} + V_{S,E}$

⑤ $\tan\theta = \dfrac{12}{5}$

$13\sin\theta - \dfrac{25}{2}\cos\alpha$

$= 13 \cdot \dfrac{12}{13} - \dfrac{25}{2} \cdot \dfrac{24}{25}$

$= 12 - 12$

$= 0$

$\dfrac{25}{2}\sin\alpha - 13\cos\theta$

$= \dfrac{25}{2} \cdot \dfrac{7}{25} - 13 \cdot \dfrac{5}{13}$

$= \dfrac{7}{2} - 5$

$= -\dfrac{3}{2}$

$\therefore V_{B,E} = \uparrow \dfrac{3}{2}$ km h^{-1} (To the North) ⑤

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[see page two

2. Since $AB \perp BC$ and $AD \perp DC$,
   although the impulse applied to A generates impulses
   I_1 and I_2 along the strings AB and AD respectively,
   no impulse is generated along BC or DC. ⑤
   Since the strings are inextensible, let us take
   the velocities of the particles at A and B in the direction
   BA as $\mathbf{u}$ and the velocities of the particles at A and D
   in the direction DA as $\mathbf{v}$. ⑤
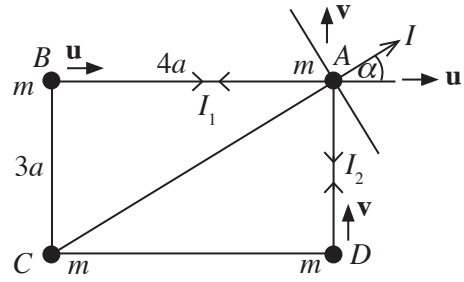
By applying $\mathbf{I} = \Delta(m\mathbf{v})$ to the system, in the direction $\overrightarrow{BA}$, we obtain $I\cos\alpha = 2mu$

By applying $\mathbf{I} = \Delta(m\mathbf{v})$ to the system, in the direction $\overrightarrow{DA}$, we obtain $I\sin\alpha = 2mv$ ⑤

$\sin\alpha = \dfrac{3}{5}$ and $\cos\alpha = \dfrac{4}{5}$

$\therefore u = \dfrac{4}{5}I \cdot \dfrac{1}{2m} = \dfrac{2I}{5m}$

$v = \dfrac{3}{5}I \cdot \dfrac{1}{2m} = \dfrac{3I}{10m}$ ⑤

$\therefore$ The speed of $A = \dfrac{I}{10m}\sqrt{4^2+3^2} = \dfrac{I}{2m}$
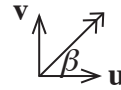
$\beta = \tan^{-1}\left(\dfrac{v}{u}\right) = \tan^{-1}\left(\dfrac{3}{4}\right) = \alpha$

The speed of $B = \dfrac{2I}{5m}$ in the direction BA

The speed of $C = 0$

The speed of $D = \dfrac{3I}{10m}$ in the direction DA ⑤

$\boxed{25}$

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3. By applying $Pv = H$ for the horizontal motion, $H = 3P$ ———— (1)
   By applying $\mathbf{F} = m\mathbf{a}$, $P - k \cdot 3^2 = 0$ ———— (2) ⑤
   $\Rightarrow H = 27k$ ———— (3)

   For the upward motion along the inclined plane,
   by applying $Pv = H$, $H = 2P_1$ ———— (4)

   By applying $\mathbf{F} = m\mathbf{a}$, $P_1 - 95\,\text{g}\sin\dfrac{\pi}{6} - k \cdot 2^2 = 0$ ———— (5) ⑤
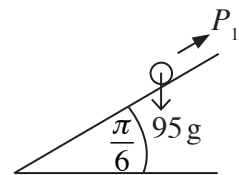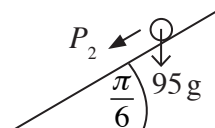

   $\Rightarrow \dfrac{H}{2} = 475 + 4k$ ———— (6)

   From (3) and (6) $19k = 950 \Rightarrow k = 50$
   $H = 1350\,\text{W} = 1.35\,\text{kW}$ ⑤

   For the downward motion along the inclined plane,
   by applying $Pv = H$, $H = 4P_2$

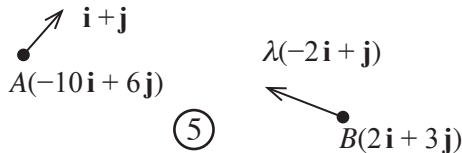   By applying $\mathbf{F} = m\mathbf{a}$, $P_2 + 95\,\text{g}\sin\dfrac{\pi}{6} - 50 \cdot 4^2 = 95a$ ⑤

   $\dfrac{1350}{4} + 475 - 800 = 95a$

   $a = \dfrac{1}{2\times 95}(675 - 650)$

   $= \dfrac{25}{2\times 95} = \dfrac{5}{38}\,\text{m s}^{-2}$ ⑤

$\boxed{25}$

4. When $t = 0$

$A(-10\mathbf{i} + 6\mathbf{j})$ with vector $\mathbf{i} + \mathbf{j}$ ⑤

$\lambda(-2\mathbf{i} + \mathbf{j})$ $B(2\mathbf{i} + 3\mathbf{j})$

If the two particles collide when $t = T$,
the displacement of $P = (\mathbf{i} + \mathbf{j})T$
the displacement of $Q = (-2\mathbf{i} + \mathbf{j})\lambda T$
By considering the position of the point at which
the collision occurs

$$-10\mathbf{i} + 6\mathbf{j} + (\mathbf{i} + \mathbf{j})T \equiv 2\mathbf{i} + 3\mathbf{j} + (-2\mathbf{i} + \mathbf{j})\lambda T$$

⑤   ⑤

$\Rightarrow (-10 + T - 2 + 2\lambda T)\mathbf{i} = (3 + \lambda T - 6 - T)\mathbf{j}$

Since $\mathbf{i} \neq \mathbf{j}$ and $\mathbf{i}, \mathbf{j} \neq \mathbf{0}$

$2\lambda T + T = 12$ and $\lambda T - T = 3$

$\Rightarrow 3T = 6$

$\Rightarrow T = 2$ $\therefore 2\lambda = 5$

$$\lambda = \frac{5}{2} \quad ⑤$$

$$\therefore \mathbf{v} = \frac{5}{2}(-2\mathbf{i} + \mathbf{j}) \Rightarrow |\mathbf{v}| = \frac{5}{2}\sqrt{(-2)^2 + 1^2} = \frac{5\sqrt{5}}{2} \quad ⑤$$

$\boxed{25}$

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5. $\overrightarrow{OA} = \mathbf{a} + 2\mathbf{b}$, $\overrightarrow{OB} = 3\mathbf{a} - \mathbf{b}$

$OA \perp OB \quad \Rightarrow \overrightarrow{OA} \cdot \overrightarrow{OB} = 0$ ⑤

$\Rightarrow (\mathbf{a} + 2\mathbf{b}) \cdot (3\mathbf{a} - \mathbf{b}) = 0$

$\Rightarrow 3|\mathbf{a}|^2 + 5\,\mathbf{a} \cdot \mathbf{b} - 2|\mathbf{b}|^2 = 0$, ⑤ since $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

$\Rightarrow \mathbf{a} \cdot \mathbf{b} = \frac{2}{5}|\mathbf{b}|^2 - \frac{3}{5}|\mathbf{a}|^2$ ⑤

If $|\mathbf{a}| = 2$ and $|\mathbf{b}| = 1$, $\mathbf{a} \cdot \mathbf{b} = \frac{2}{5} - \frac{12}{5} = -2$

$|\mathbf{a}|\,|\mathbf{b}| \cos\theta = -2$ ⑤   Here θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.
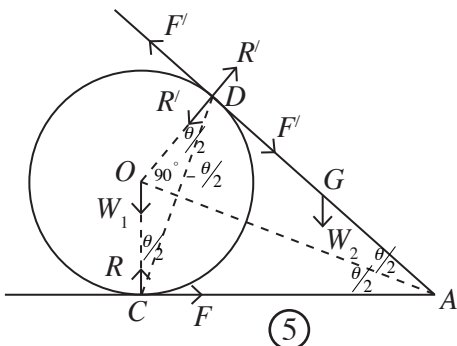
$2\cos\theta = -2$ $0 \leq \theta \leq \pi$

$\cos\theta = -1$

$\theta = \pi$ ⑤

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6. The three forces being concurrent. ⑤

Since the weight of the cylinder and the force applied on the cylinder by the rough horizontal plane pass through the point C, the force applied on the cylinder at the point D by the rod should also pass through C. ⑤

Then, $O\hat{C}D = O\hat{D}C = \dfrac{\theta}{2}$

For the equilibrium of the cylinder $\left|\dfrac{F}{R}\right| \leq \mu$. ⑤

$\therefore \tan\dfrac{\theta}{2} \leq \tan\lambda$

$\Rightarrow \lambda \geq \dfrac{\theta}{2}$, (Since $0 < \lambda$, $\theta < \dfrac{\pi}{2}$)
⑤

$\boxed{25}$

[see page four

7. Let A, B and C be respectively the events of the three children A, B and C independently solving the problem correctly.

Then $P(A) = \dfrac{1}{6}$, $P(B) = \dfrac{1}{2}$ and $P(C) = \dfrac{1}{3}$.

Let X be the event of exactly two children independently solving the problem correctly.

$$
\begin{aligned}
\text{Then, } X \quad &= \quad (A\cap B\cap C')\cup(A\cap B'\cap C)\cup(A'\cap B\cap C) \\
P(X) \quad &= \quad P\,(A\cap B\cap C')+P\,(A\cap B'\cap C)+P\,(A'\cap B\cap C) \ (\text{Axiom}) \ \text{⑤} \\
&= \quad P(A)\,P(B)\,P(C') + P(A)\,P(B')\,P(C) + P(A')\,P(B)\,P(C) \quad \text{⑤} \\
&\qquad\qquad\qquad (\text{Since } A, B, C \text{ are independent of each other}) \\
&= \quad \frac{1}{6}\,\frac{1}{2}\,(1-\frac{1}{3}) + \frac{1}{6}\,(1-\frac{1}{2})\frac{1}{3} \; + (1-\frac{1}{6})\,\frac{1}{2}\,\frac{1}{3} \ \text{⑩} \\
&= \quad \frac{1}{6}\,\frac{1}{2}\,\frac{2}{3} + \frac{1}{6}\,\frac{1}{2}\,\frac{1}{3} \; + \frac{5}{6}\,\frac{1}{2}\,\frac{1}{3} \\
&= \quad \frac{8}{36} \quad \text{⑤} \\
&= \quad \frac{2}{9}
\end{aligned}
$$

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8. R : Obtaining a plant with pink flowers

$P(R) = \dfrac{1}{6}$ and $P(R') = 1 - \dfrac{1}{6} \ = \ \dfrac{5}{6}$ 　　⑤

n : Number of seeds that need to be planted

X : Obtaining at least one plant with pink flowers

$P(X) \ = 1 - P\,(\text{Not obtaining even one plant with pink flowers})$

$$= 1 - P\,(R' \cap R' ... \cap R') \qquad\qquad \text{⑤}$$

$$= 1 - \{P\,(R')\}^{n} > 0.98 \qquad\qquad \text{⑤}$$

$$0.02 > \left(\frac{5}{6}\right)^{n}$$

$$n > \frac{\ln(0.02)}{\ln\left(\dfrac{5}{6}\right)} = 21.46 \qquad\qquad \text{⑤}$$

$$\therefore \ n_{minimum} = 22 \qquad\qquad\qquad\qquad \text{⑤}$$

$\boxed{25}$

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9.

x	-2	-1	0	1	2
f	4	1	3	1	1
$f \cdot x$	-8	-1	0	1	2
$f \cdot x^2$	16	1	0	1	4

⑤

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{-6}{10} = -0.6 \quad ⑤$$

$$\sigma_x^2 = \frac{\sum fx^2}{\sum f} - \bar{x}^2 = \frac{22}{10} - 0.36 = 1.84 \quad ⑤$$

$$\therefore \sigma_x = \sqrt{1.84}$$

Let $y = 2000 - 4x$

Then $\bar{y} = 2000 - 4\bar{x}$

$= 2000 + 2.4$

$= 2002.4 \quad ⑤$

$$\sigma_y^2 = 4^2 \cdot \sigma_x^2$$

$$= 16 \times 1.84$$

$$\therefore \sigma_y = 4\sqrt{1.84} \quad ⑤$$

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10.

$$\bar{x} = \frac{\sum x_i}{20} = 40$$

$\therefore$ The sum of all the marks $= 20 \times 40 = 800$

The sum of the least **six** marks $= 6 \times 25 = 150 \quad ⑤$

The sum of the highest **six** marks $= 440$

$\therefore$ The sum of the marks of the remaining 8 children

$$= 800 - 590 = 210 \quad ⑤$$

(i) $\therefore$ The mean of the marks of the remaining 8 children

$$= \frac{210}{8} = 26.25 \quad ⑤$$

(ii) The third quartile $= Q_3$

$$\frac{3}{4}(n+1) = \frac{3}{4} \times 21 = 15\frac{3}{4}$$

The 15th mark $= 70 \quad ⑤$

The 16th mark $= 71$

$$\therefore Q_3 = 70 + \frac{3}{4}(71 - 70)$$

$$= 70.75 \quad ⑤$$

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11.　(a)

$\tan \beta = g$

⑤　⑩　⑩
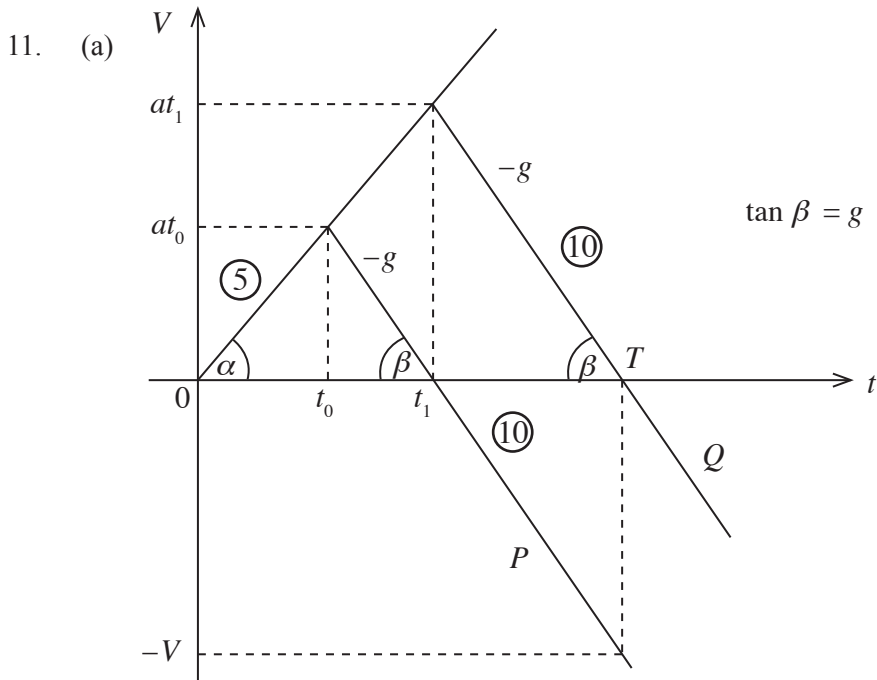
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For the motion of the particle P,

$\tan \beta = \dfrac{at_0}{t_1 - t_0}$ ⑤　⟹　$g = \dfrac{at_0}{t_1 - t_0}$ ⑤

$t_1 - t_0 = \dfrac{at_0}{g}$

$t_1 = \dfrac{at_0}{g} + t_0$

$t_1 = \dfrac{t_0}{g}(a + g)$ ⑤

If the time taken for Q to come to instantaneous rest is T,

$\tan \beta = \dfrac{at_1}{T - t_1}$ 　⟹　$g = \dfrac{at_1}{T - t_1}$ ⟹ $T - t_1 = \dfrac{at_1}{g}$ ⑤

$\tan \beta = \dfrac{V}{T - t_1}$

$V = g(T - t_1)$

$= \cancel{g}\,\dfrac{at_1}{\cancel{g}} = at_0\left(\dfrac{a}{g} + 1\right)$ ⑤

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(b)
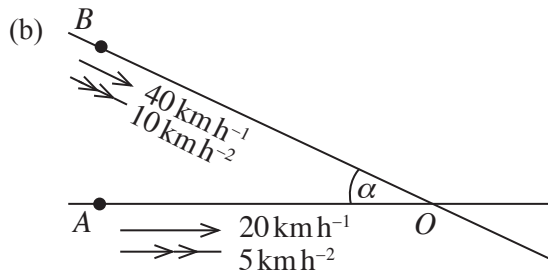
$$V(Y, X) = V(Y, E) + V(E, X)$$

$$= \text{---} \overset{\alpha}{\searrow} \, 40\,\text{km}\,\text{h}^{-1} + \longleftarrow 20\,\text{km}\,\text{h}^{-1} \quad \text{⑤}$$

$$= \overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$$
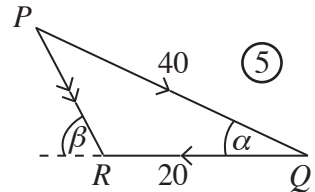
and $a(Y, X) = a(Y, E) + a(E, X)$

$$= \overset{10\,\text{km}\,\text{h}^{-2}}{\underset{\alpha}{\searrow}} + \longleftarrow \underset{5\,\text{km}\,\text{h}^{-2}}{} \quad \text{⑤}$$

$$= \overrightarrow{LN} + \overrightarrow{NM} = \overrightarrow{LM}$$

(i) $\quad \tan \beta = \dfrac{40 \sin \alpha}{40 \cos \alpha - 20} \quad \text{⑤}$

$$= \dfrac{2 \sin \alpha}{2 \cos \alpha - 1}$$
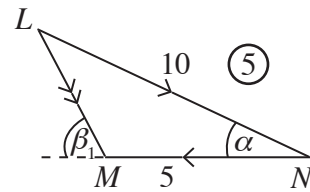
(ii) $\quad \tan \beta_1 = \dfrac{10 \sin \alpha}{10 \cos \alpha - 5} = \dfrac{2 \sin \alpha}{2 \cos \alpha - 1} \quad \text{⑤}$

From (i) and (ii), $\tan \beta = \tan \beta_1$

$$\beta = \beta_1 \; (0 < \beta, \beta_1 < \pi) \quad \text{⑤}$$

$\Rightarrow V(Y, X) \, \nearrow\!\!\!\nearrow \, a(Y, X) \quad \text{⑤}$

$\therefore$ The path of Y relative to X is a straight line.

$\boxed{40}$

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If $\alpha = 60°$, then $\tan \beta$, $\tan \beta_1$ are not defined.

$$\therefore \beta = \beta_1 = 90° \quad \text{⑤}$$

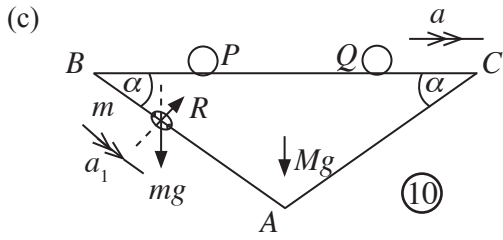$\therefore$ The path of Y relative to X is perpendicular to OA.
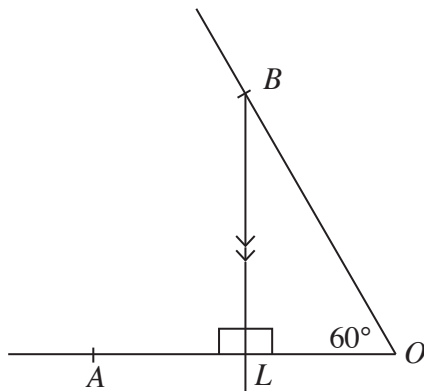
The shortest distance between X and Y

$$= AL = 10 - 10 \cos 60°$$

$$= 5\,\text{km} \quad \text{⑤}$$

$\boxed{10}$

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(c)

$$a\,(M, E) \;=\; \longrightarrow a \;, \quad a\,(m, E) \;= \;\text{(vector diagram)} \quad ⑤$$

By applying $\mathbf{F} = m\mathbf{a}$

$$m : \text{(angle } \alpha) \;; \quad mg\sin\alpha \;=\; m\,(a_1 + a\cos\alpha) \;\text{———— (1)} \quad ⑤$$

$$M, m \longrightarrow ; \qquad 0 \;=\; Ma + m\,(a + a_1\cos\alpha)$$

$$0 \;=\; (M + m)\,a + m\,a_1\cos\alpha \;\text{———— (2)} \quad ⑩$$

From (1) and (2) ;

$$0 \;=\; (M + m)\,a + m\,(g\sin\alpha - a\cos\alpha)\cos\alpha \quad ⑤$$

$$- mg\sin\alpha\cos\alpha \;=\; (M + m - m\cos^2\alpha)\,a$$

$$\therefore a \;=\; -\frac{mg\sin\alpha\cos\alpha}{M + m\sin^2\alpha} \;=\; -\frac{mg\sin 2\alpha}{2(M + m\sin^2\alpha)}$$

$$⑤$$

$$m : \text{(angle } \alpha, R) \;; \quad R - mg\cos\alpha \;=\; ma\sin\alpha \quad ⑤$$

$$R \;=\; mg\cos\alpha - \frac{m^2 g\sin^2\alpha\cos\alpha}{M + m\sin^2\alpha}$$

$$=\; \frac{mg\cos\alpha}{M + m\sin^2\alpha}\,\{M + m\sin^2\alpha - m\sin^2\alpha\} \quad ⑤$$

$$=\; \frac{Mmg\cos\alpha}{M + m\sin^2\alpha}$$

$$\boxed{50}$$

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12. (a)

(i)
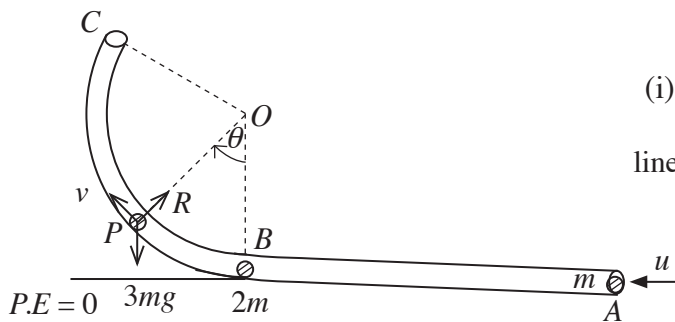
By the law of conservation of linear momentum;

$$3mv_1 \;=\; mu \quad ⑤$$

$$v_1 \;=\; \frac{u}{3} \quad ⑤$$

$$\boxed{10}$$

....

(ii) By the law of conservation of energy

$$\frac{1}{2}(3m)v^2 + 3mg(a - a\cos\theta) = \frac{1}{2}(3m)\frac{u^2}{9} \qquad \text{⑮}$$

$$v^2 = \frac{u^2}{9} - 2ga(1 - \cos\theta) \qquad \text{⑤}$$

Applying $\quad$ **F** $=$ m**a**

$$\overrightarrow{PO};\ R - 3mg\cos\theta = 3m\left(\frac{v^2}{a}\right) \qquad \text{⑩}$$

$$R = \frac{3m}{a}\left\{\frac{u^2}{9} - 2ga + 3ga\cos\theta\right\} \qquad \text{⑤} \qquad \boxed{35}$$

....

(iii) When $\theta = \frac{2\pi}{3};\quad v^2 = \frac{u^2}{9} - 2ga\left(1 + \frac{1}{2}\right) = \frac{u^2}{9} - 3ga \qquad \text{⑤}$

⑤ If $\frac{u^2}{9} - 3ga > 0$ the particle leaves at C ; i.e., when $u > 3\sqrt{3ga}$ $\qquad \boxed{10}$

....

(iv) Applying $s = ut + \frac{1}{2}at^2$ for the motion from C to A,

$$\rightarrow \sqrt{3}\,a + \frac{\sqrt{3}a}{2} = v\left(\frac{1}{2}\right)t \quad \Rightarrow 3\sqrt{3}\,a = vt \text{ ——— (1)} \qquad \text{⑤}$$

$$\uparrow -\frac{3a}{2} = v\frac{\sqrt{3}}{2}t - \frac{1}{2}gt^2 \text{ ——— (2)} \qquad \text{⑤}$$

From (1) and (2),

$$-\frac{3a}{2} = v\frac{\sqrt{3}}{2} \cdot \frac{3\sqrt{3}a}{v} - \frac{1}{2}g\frac{27}{v^2}a^2 \qquad \text{⑤}$$

$$\frac{1}{2}ga\frac{27}{v^2} = \frac{9}{2} + \frac{3}{2}$$

$$27ga = 12v^2$$

$$\frac{9}{4}ga = \frac{u^2}{9} - 3ga$$

$$\frac{u^2}{9} = \frac{9}{4}ga + 3ga \qquad \text{⑤}$$

$$\therefore u = \frac{3}{2}\sqrt{21ag}$$

$$\boxed{20}$$

....

(b) By the law of conservation of linear momentum

$$mv_1 + Mv_2 = mu \text{ ——— (i)} \qquad \text{⑩}$$

By Newton's law of restitution $\quad v_1 - v_2 = -eu$ ——— (ii) $\qquad \text{⑩}$

From (i) and (ii) $\quad v_1 = \frac{u(m - eM)}{(m + M)}$ and $v_2 = \frac{mu(1 + e)}{(m + M)}$ $\qquad \text{⑩}$

[see page ten

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Applying $\Delta E = \sum \frac{1}{2} I.(U+V)$ ⑤

$\quad = \frac{1}{2}(I)(v_2) + \frac{1}{2}(-I)(u+v_1) = -\frac{I}{2}(u+v_1-v_2) = -\frac{I}{2}(u-eu)$ ⑩

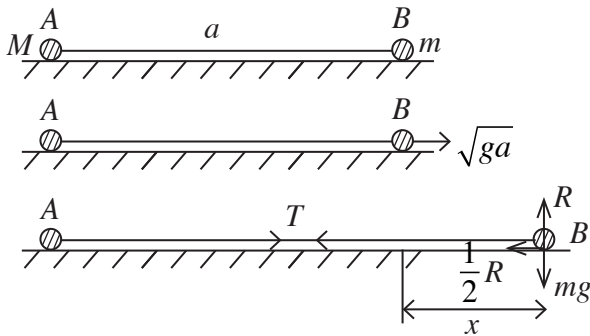$\quad = -\frac{I}{2}(1-e)u$ ⑤ ; where $I = Mv_2 = \dfrac{Mmu(1+e)}{(M+m)}$ ⑤

$\dfrac{1}{2}\dfrac{Mmu^2(1+e)(1-e)}{(M+m)} = \dfrac{1}{4}mu^2$ ⑩

$M(1-2e^2) = m$ ⑤ $\Rightarrow$ $1-2e^2 > 0$ ⑤ $\Rightarrow$ $e < \dfrac{1}{\sqrt{2}}$

$\boxed{75}$

13.

By applying $\uparrow$ $\mathbf{F} = m\mathbf{a}$ to the particle B, $R = mg$

By applying $\rightarrow$ $\mathbf{F} = m\mathbf{a}$ to the particle B, $-\frac{1}{2}R - T = m\ddot{x}$ ⑩

Since the modulus of elasticity of the string is $2mg$, $T = 2mg\dfrac{x}{a}$ ⑩

$\Rightarrow$ $-\dfrac{1}{2}mg - \dfrac{2mgx}{a} = m\ddot{x}$ ⑩

$\Rightarrow$ $\ddot{x} = -\dfrac{2g}{a}\left(x + \dfrac{a}{4}\right)$ ———— (1)

$\therefore$ the motion of the particle is a simple harmonic motion with the centre of oscillation given by $x = -\dfrac{a}{4}$ ⑤

By taking $x + \dfrac{a}{4} = \alpha\cos\omega t + \beta\sin\omega t$, ———— (2)

since $x = 0$ when $t = 0$, $\alpha = \dfrac{a}{4}$ ⑤

By differentiating with respect to t, $\dot{x} = -\alpha\omega\sin\omega t + \beta\omega\cos\omega t$ ———— (3)

Since $\dot{x} = \sqrt{ga}$ when $t = 0$, $\sqrt{ga} = \beta\omega$ $\Rightarrow$ $\beta = \dfrac{\sqrt{ga}}{\omega}$ ⑤

By differentiating again with respect to t, $\ddot{x} = -\alpha\omega^2\cos\omega t - \beta\omega^2\sin\omega t$
$= -\omega^2(\alpha\cos\omega t + \beta\sin\omega t)$
$= -\omega^2(x + \dfrac{a}{4})$ ⑤

From (1), $\omega^2 = \dfrac{2g}{a} \Rightarrow \omega = \sqrt{\dfrac{2g}{a}}$ ⑤

$\therefore \alpha = \dfrac{a}{4}$ and $\beta = \sqrt{ga}\,\dfrac{\sqrt{a}}{\sqrt{2g}} = \dfrac{a}{\sqrt{2}}$ ⑤

$\boxed{60}$

Since $\dot{x} = 0$ at the maximum extension, ⑩
from (3), $\alpha \sin \omega t = \beta \cos \omega t$

$\Rightarrow \tan \omega t = \dfrac{\beta}{\alpha} = \dfrac{a/\sqrt{2}}{a/4} = 2\sqrt{2}$ ⑤

From (2), $x + \dfrac{a}{4} = \dfrac{a}{4}\,\dfrac{1}{3} + \dfrac{a}{\sqrt{2}}\,\dfrac{2\sqrt{2}}{3}$

$= \dfrac{a}{12}\,(1 + 8)$

$= \dfrac{3a}{4}$

$\Rightarrow \quad x = \dfrac{a}{2}$ ⑤

$\therefore$ The maximum extension is $\dfrac{a}{2}$.

$\boxed{20}$

The maximum tension $= \dfrac{2mg}{a}\,\dfrac{a}{2} = mg$ ⑤

For the equilibrium of particle A,
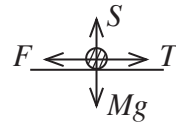
$S = Mg$ and $T = F$

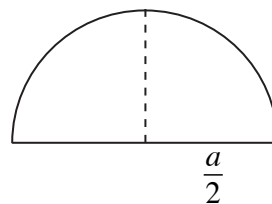Since $\dfrac{F}{S} \le \dfrac{1}{2}$, $\dfrac{mg}{Mg} \le \dfrac{1}{2}$ ⑤
⑩

$\therefore \quad M \ge 2m$

$\boxed{20}$
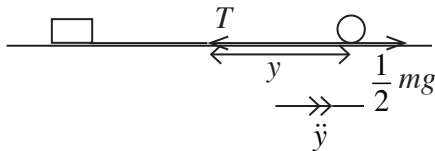
$-T + \dfrac{1}{2}\,mg = m\ddot{y}$ ⑤

$-\dfrac{2mg}{a}\,y + \dfrac{mg}{2} = m\ddot{y}$

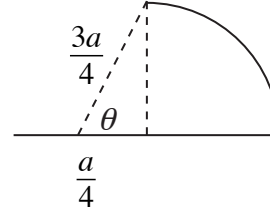$\ddot{y} = -\dfrac{2g}{a}\left(y - \dfrac{a}{4}\right)$ ⑤

$\boxed{10}$

Centre of oscillation $= \dfrac{a}{4}$ ⑤

Angular velocity $= \sqrt{\dfrac{2g}{a}}$ ⑤

By considering the circular motion corresponding to the simple harmonic motion :

Time taken to return $= \pi\sqrt{\dfrac{a}{2g}}$ ⑤

Time taken to reach maximum extension

$= \dfrac{\theta}{\omega} = \sqrt{\dfrac{a}{2g}}\cos^{-1}\dfrac{1}{3}$ ⑤

$\therefore$ Total time $= \sqrt{\dfrac{a}{2g}}\cos^{-1}\dfrac{1}{3} + \pi\sqrt{\dfrac{a}{2g}}$

$= \left[\pi + \cos^{-1}\left(\dfrac{1}{3}\right)\right]\sqrt{\dfrac{a}{2g}}$ ⑤

For the return motion, $\dot{y} = \dfrac{a}{4}\left(-\sqrt{\dfrac{2g}{a}}\right)\sin\sqrt{\dfrac{2g}{a}}\,t$ ⑤

When $t = \pi\sqrt{\dfrac{a}{2g}}$

$\sqrt{\dfrac{2g}{a}}\,t = \pi$

$\therefore \dot{y} = 0$ ⑤

$\therefore$ The particle B returns to the initial point and comes to a definite rest. ⑤

$\boxed{40}$

....

Aliter :

For the motion of particle B :

Let the time for particle B to reach the maximum extension be t_0.

When $\tan \omega t = 2\sqrt{2}$, then $\cos \omega t = \dfrac{1}{3}$ ⑤

$\therefore \cos \omega t_0 = \dfrac{1}{3}$

$\omega t_0 = \cos^{-1}\left(\dfrac{1}{3}\right)$

$t_0 = \dfrac{1}{\omega}\cos^{-1}\left(\dfrac{1}{3}\right) = \sqrt{\dfrac{a}{2g}}\cos^{-1}\left(\dfrac{1}{3}\right)$ ⑤

For the return motion of particle B :

Let the time taken to reach the initial point be t_1.

Then $y = 0$. ⑤

$\therefore \cos\sqrt{\dfrac{2g}{a}}\,t_1 = -1$

[see page thirteen

$$\sqrt{\frac{2g}{a}}\, t_1 \;=\; \pi \quad\Rightarrow\quad t_1 \;=\; \sqrt{\frac{a}{2g}}\,\pi \qquad ⑤$$

$\therefore$ The time taken for particle B to reach the initial point

$$= \; t_0 + t_1$$

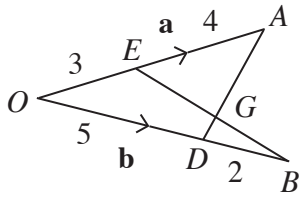$$= \; \sqrt{\frac{a}{2g}}\left[\pi + \cos^{-1}\!\left(\frac{1}{3}\right)\right] \qquad ⑤$$

If the velocity when $t_1 = \sqrt{\dfrac{a}{2g}}\,\pi$ is $\dot{y}$,

$$\dot{y} \;=\; -\frac{a}{4}\sqrt{\frac{2g}{a}}\;\sin\sqrt{\frac{2g}{a}}\,t_1 \quad ⑤$$

$$= \; -\frac{a}{4}\sqrt{\frac{2g}{a}}\;\sin\sqrt{\frac{2g}{a}}\,\sqrt{\frac{a}{2g}}\,\pi$$

$$= \; 0 \qquad ⑤$$

$\therefore$ Particle B comes to rest definitely at the initial point ⑤

$\boxed{40}$

....

14. (a)

$$\overrightarrow{OG} \;=\; \overrightarrow{OB} + \overrightarrow{BG}$$
$$= \; \overrightarrow{OB} + \lambda\,\overrightarrow{BE}$$
$$= \; \overrightarrow{OB} + \lambda[\,\overrightarrow{BO} + \overrightarrow{OE}\,]$$
$$= \; \overrightarrow{OB} + \lambda(-\overrightarrow{OB} + \frac{3}{7}\overrightarrow{OA})$$
$$= \; \mathbf{b} + \lambda(\frac{3}{7}\mathbf{a} - \mathbf{b}) \;\text{———}\;(1) \qquad ⑮$$

$$\overrightarrow{OG} \;=\; \overrightarrow{OA} + \overrightarrow{AG}$$
$$= \; \overrightarrow{OA} + \mu\,\overrightarrow{AD}$$
$$= \; \overrightarrow{OA} + \mu[\,\overrightarrow{AO} + \overrightarrow{OD}\,]$$
$$= \; \overrightarrow{OA} + \mu(-\overrightarrow{OA} + \frac{5}{7}\overrightarrow{OB})$$
$$= \; \mathbf{a} + \mu(\frac{5}{7}\mathbf{b} - \mathbf{a}) \;\text{———}\;(2) \qquad ⑮$$

From (1) and (2),

$$\mathbf{b} + \lambda(\frac{3}{7}\mathbf{a} - \mathbf{b}) \;=\; \mathbf{a} + \mu(\frac{5}{7}\mathbf{b} - \mathbf{a}) \qquad ⑤$$

Since $\mathbf{a}$ and $\mathbf{b}$ are independent vectors,

$$\frac{3\lambda}{7} \;=\; 1 - \mu \;\text{———}\;(3) \qquad ⑤$$

$$1 - \lambda \;=\; \frac{5\mu}{7} \;\text{———}\;(4) \qquad ⑤$$

$(3) + \frac{3}{7}\times(4)$

$$\frac{3}{7} \;=\; 1 - \mu + \frac{15}{49}\mu$$

$$\frac{\cancel{4}^{\,2}}{\cancel{7}\,1} \;=\; \frac{17}{7}\,\frac{\cancel{34}}{\cancel{49}}\,\mu$$

$$\Rightarrow \quad \mu = \frac{14}{17} \qquad \text{⑤}$$

$$\overrightarrow{OG} = \mathbf{a} + \frac{14}{17}\left(\frac{5}{7}\mathbf{b} - \mathbf{a}\right) = \mathbf{a} + \frac{10}{17}\mathbf{b} - \frac{14}{17}\mathbf{a}$$

$$= \frac{3}{17}\mathbf{a} + \frac{10}{17}\mathbf{b} = \frac{1}{17}(3\mathbf{a} + 10\mathbf{b}) \qquad \text{⑩} \qquad \boxed{60}$$

....

(b)
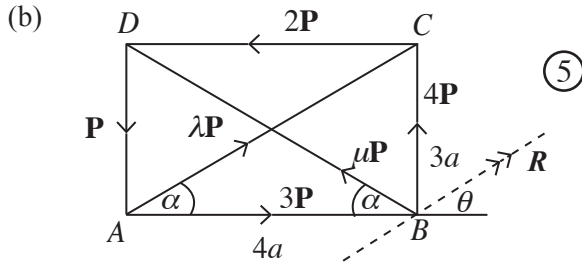
(i) $R = 0$ when the system is equivalent to a couple. ⑤

$$\text{Then} \longrightarrow X = 0, \quad \uparrow Y = 0.$$

$\longrightarrow X$:
$$3P + \lambda P\cos\alpha - \mu P\cos\alpha - 2P = 0 \qquad \text{⑤}$$
$$3P + \lambda P\frac{4}{5} - \mu P\frac{4}{5} - 2P = 0$$
$$5\cancel{P} + 4\lambda\cancel{P} - 4\mu\cancel{P} = 0$$
$$5 = 4\mu - 4\lambda \quad \text{———— (1)} \quad \text{⑤}$$

$\uparrow Y$:
$$4P - P + \lambda P\sin\alpha + \mu P\sin\alpha = 3P + \lambda P\frac{3}{5} + \mu P\frac{3}{5} = 0 \qquad \text{⑤}$$
$$\lambda + \mu = -5 \quad \text{———— (2)} \quad \text{⑤}$$

$(1) + (2) \times 4$

$$8\mu = -15 \Rightarrow \mu = -\frac{15}{8} \qquad \text{⑤}$$
$$\lambda = -5 - \mu = -5 + \frac{15}{8} = -\frac{25}{8} \qquad \text{⑤} \qquad \boxed{40}$$

....

(ii) $\curvearrowleft \widehat{B} = 0$ ⑤

$$2\cancel{P} \times 3\cancel{a} + \cancel{P} \times 4\cancel{a} - \lambda\cancel{P} \times 4\cancel{a}\sin\alpha = 0$$
$$6 + 4 - \lambda \times 4 \times \frac{3}{5} = 0 \Rightarrow \lambda = \frac{\cancel{50}^{25}}{\cancel{12}_6} = \frac{25}{6} \qquad \text{⑤}$$

$$\longrightarrow R\cos\theta = P + (\lambda - \mu)P\cos\alpha \quad \text{———— (1)} \qquad \text{⑤}$$
$$\uparrow R\sin\theta = 3P + (\lambda + \mu)P\sin\alpha \quad \text{———— (2)} \qquad \text{⑤}$$

From (2)/(1), $\quad \tan\theta = \dfrac{3P + (\lambda + \mu)P \times \frac{3}{5}}{P + (\lambda - \mu)P \times \frac{4}{5}}$ ⑤

[see page fifteen

Since R is parallel to AC, $\quad \tan\theta = \tan\alpha$

$$\theta = \alpha \qquad \text{⑤}$$

$$\frac{3}{4} \, \text{⑤} = \frac{3+\left(\frac{25}{6}+\mu\right)\times\frac{3}{5}}{1+\left(\frac{25}{6}-\mu\right)\times\frac{4}{5}} = \frac{3+\frac{5}{2}+\frac{3\mu}{5}}{1+\frac{5\times2}{3}-\frac{4\mu}{5}}$$
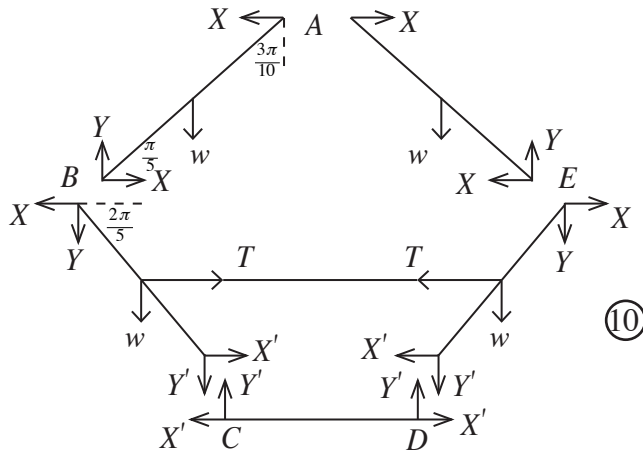
$$13-\frac{12\mu}{5} = 22+\frac{12\mu}{5} \implies 9 = -\frac{24\mu}{5} \implies \mu = -\frac{45}{24} \quad \text{⑤}$$

For equilibrium, $\boldsymbol{R} = 0$ and the moment about any point should be zero.
From parts (i) and (ii), it is clear that it is not possible to find λ, μ satisfying both these conditions simultaneously. $\quad$ ⑩

$$\boxed{50}$$

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15. (a)

Let the length of a rod be $2a$. The system is symmetric about the vertical line through A.
$\therefore Y = 0$ at A. ⑤

For the equilibrium of the rod AB,

$\overset{\curvearrowleft}{B}\rangle$ $\quad X\times 2a\sin\frac{\pi}{5} - w\times a\cos\frac{\pi}{5} = 0$ $\qquad$ ⑩

$$X = \frac{w}{2}\cot\frac{\pi}{5} \qquad\qquad \text{⑤}$$

For the equilibrium of the rod $AB \uparrow$,

$\uparrow$ $\quad Y = w$ $\quad$ ⑤

For the equilibrium of the rod BC,

$\overset{\curvearrowleft}{C}\rangle$ $\quad X\times 2a\sin\frac{2\pi}{5} + Y\times 2a\cos\frac{2\pi}{5} + w\times a\cos\frac{2\pi}{5} - T\times a\sin\frac{2\pi}{5} = 0$ $\qquad$ ⑩
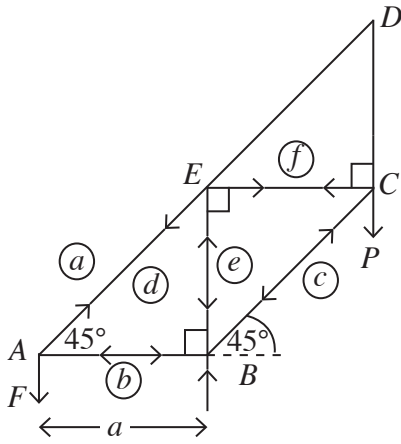
$$T = 2X + Y\,2\cot\frac{2\pi}{5} + w\cot\frac{2\pi}{5}$$

$$= 2\!\!\!\diagup\,\frac{w}{2\!\!\!\diagup}\cot\frac{\pi}{5} + 2w\cot\frac{2\pi}{5} + w\cot\frac{2\pi}{5} \quad \text{⑤}$$

$$= w\left(\cot\frac{\pi}{5} + 3\cot\frac{2\pi}{5}\right) \qquad\qquad \text{⑩}$$
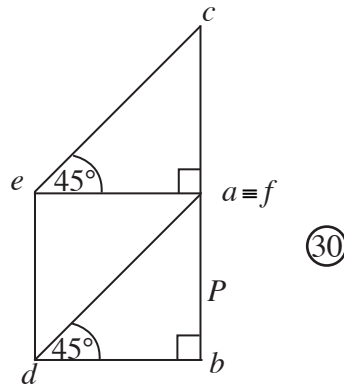
$$\boxed{60}$$

....

(b)

$\overset{\curvearrowleft}{B}$; $F \times a - P \times a = 0$

$F = P\,\text{N}$ ⑩

$a \equiv f$

a and f are concurrent. ⑤

㉚

Rod	Magnitude	Stress
AB	P N	Thrust
BC	$\sqrt{2}\,P$ N	Thrust
CD	0	–
DE	0	–
AE	$\sqrt{2}\,P$ N	Tension
BE	P N	Thrust
EC	P N	Tension

㉟

The reaction on the support $\boldsymbol{R} = 2P$ N ⑩

90

....

16.
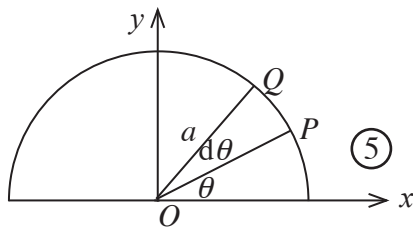
Due to symmetry about Oy, the centre of mass lies on Oy. ⑤

Let ρ be the mass of a unit length. The radius of the wire frame is a.

The mass of the elemental arc $PQ = \rho\, a\, d\theta$

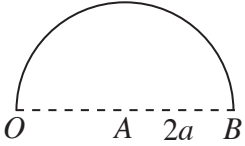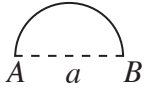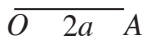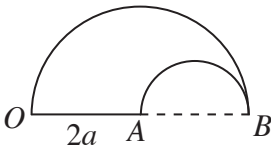The distance from Ox to the centre of mass of PQ is $y = a\sin\theta$

If the distance from O to the centre of mass of the object is $\bar{y}$,

$$\bar{y} = \frac{\int y\, dm}{\int dm}$$

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$$= \frac{\int_0^\pi \rho a \sin\theta \, a \, \mathrm{d}\theta}{\int_0^\pi \rho a \, \mathrm{d}\theta} \; \text{⑩} \quad = \quad \frac{a^2 \rho \int_0^\pi \sin\theta \, \mathrm{d}\theta}{a\rho \int_0^\pi \mathrm{d}\theta}$$

$$= \frac{a^2 \rho \left[-\cos\theta\right]_0^\pi}{a\rho \left[\theta\right]_0^\pi} \quad = \quad \frac{a\left[-\cos\pi + \cos 0\right]}{\pi} \quad \text{⑩}$$

$$= \frac{2a}{\pi}$$

[30]

....

Let ρ be the mass of a unit length.

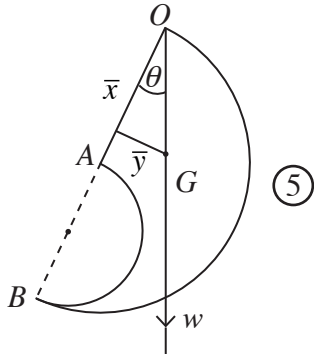| Object	Mass	Distance from the x axis	Distance from the y axis
	$2\pi a \rho$	$\dfrac{4a}{\pi}$	$2a$
	$\pi a \rho$	$\dfrac{2a}{\pi}$	$3a$
	$2a\rho$	0	a
	$3\pi a \rho + 2a\rho$	$\overline{y}$	$\overline{x}$

⑩

$$\overline{x} \quad = \quad \frac{2a \times 2\pi a\rho + 3a \times \pi a\rho + 2a\rho \times a}{3\pi a\rho + 2a\rho} \qquad \text{⑩}$$

$$= \quad \frac{4\pi a + 3\pi a + 2a}{3\pi + 2} \quad = \quad \frac{7\pi a + 2a}{3\pi + 2} \qquad \text{⑤}$$

$$\overline{y} \quad = \quad \frac{\dfrac{4a}{\pi} \times 2\pi a\rho + \dfrac{2a}{\pi} \times \pi a\rho}{3\pi a\rho + 2a\rho} \text{⑩} \quad = \quad \frac{10a}{3\pi + 2} \qquad \text{⑤}$$

$$G(\overline{x}, \overline{y}) = \quad \left[\frac{(7\pi + 2)a}{3\pi + 2}, \; \frac{10a}{3\pi + 2}\right]$$

[60]

....

$$\tan \theta = \frac{\overline{y}}{\overline{x}} = \frac{\dfrac{10a}{3\pi+2}}{\dfrac{(7\pi+2)a}{3\pi+2}} \; ⑩ = \frac{10}{7\pi+2} \quad ⑤$$
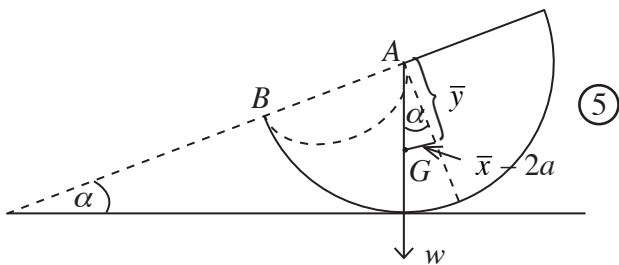
⑤

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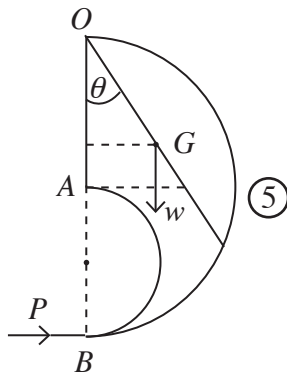
By taking moments about O,

$O\curvearrowleft$;

$$P \times 4a = w \times \overline{y} \quad ⑩$$
$$P = \frac{w}{4a} \times \frac{10a}{3\pi+2}$$
$$= \frac{5w}{2(3\pi+2)} \; ⑤ \text{ units}$$

⑤

20

$$\tan \alpha = \frac{\overline{x}-2a}{\overline{y}}$$
$$= \frac{\dfrac{(7\pi+2)a}{3\pi+2}-2a}{\dfrac{10a}{3\pi+2}} \; ⑩$$
$$= \frac{(7\pi+2)a-2a(3\pi+2)}{10a}$$
$$\tan \alpha = \frac{\pi-2}{10}$$
$$\alpha = \tan^{-1}\left(\frac{\pi-2}{10}\right) \; ⑤$$

⑤

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[see page nineteen

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17.   (a) $P(A|B) \;=\; \dfrac{P(A \cap B)}{P(B)} \;\;;\;\; P(B) > 0$ ⑤   ⑤

$$
\begin{aligned}
P(A_1 \cap A_2 \cap A_3) &= P[(A_1 \cap A_2) \cap A_3] \\
&= P(X \cap A_3) \;;\; \text{Here } X = A_1 \cap A_2. \\
&= P(X).P(A_3|X) \quad ⑤ \\
P(X) &= P(A_1 \cap A_2) \\
&= P(A_1).P(A_2|A_1) \quad ⑤
\end{aligned}
$$

$\therefore \; P(A_1 \cap A_2 \cap A_3) = P(A_1). \, P(A_2|A_1). \, P(A_3|A_1 \cap A_2)$ ⑩

A	:	The selected member being an adult
C	:	The selected member being a child
F	:	The selected member being a female
M	:	The selected member being a male
S	:	The selected member being one who uses the swimming pool.

$P(A) = \dfrac{3}{4}$ $P(C) = \dfrac{1}{4}$

$P(M|A) = \dfrac{3}{4}$ $P(F|A) = \dfrac{1}{4}$

$P(M|C) = \dfrac{3}{5}$ $P(F|C) = \dfrac{2}{5}$

$P(S|A \cap M) = \dfrac{1}{2}$ $P(S|A \cap F) = \dfrac{1}{3}$

$P(S|C \cap M) = \dfrac{4}{5}$ $P(S|C \cap F) = \dfrac{4}{5}$ $\Rightarrow$ $P(S|C) = \dfrac{4}{5}$ ⑩

(i)
$$
\begin{aligned}
P(S) &= P(A \cap M \cap S) + P(A \cap F \cap S) + P(C \cap S) \quad ⑤ \\
&= P(A)\,P(M|A)\,P(S|A \cap M) + P(A)\,P(F|A)\,P(S|A \cap F) + P(C)\,P(S|C) \quad ⑤ \\
&= \frac{3}{4} \times \frac{3}{4} \times \frac{1}{2} + \frac{3}{4} \times \frac{1}{4} \times \frac{1}{3} + \frac{1}{4} \times \frac{4}{5} \quad ⑤ \\
&= \frac{9}{32} + \frac{1}{16} + \frac{1}{5} \;=\; \frac{87}{160} \quad ⑤
\end{aligned}
$$
⑤   ③⓪

(ii) $P(M|S) = \dfrac{P(M \cap S)}{P(S)}$

$$
\begin{aligned}
P(M \cap S) &= P(A \cap M \cap S) + P(C \cap M \cap S) \quad ⑤ \\
&= \frac{9}{32} + \frac{1}{4} \times \frac{3}{5} \times \frac{4}{5} \\
&= \frac{321}{800} \quad ⑤
\end{aligned}
$$

$\therefore P(M|S) = \dfrac{P(M \cap S)}{P(S)} = \dfrac{\overset{107}{\cancel{321}}}{\underset{5}{\cancel{800}}} \times \dfrac{\overset{\cancel{160}}{}}{\underset{29}{\cancel{87}}} = \dfrac{107}{145}$ ⑩

(iii) $P(C \cap M | S') = \dfrac{P(C \cap M \cap S')}{P(S')}$

$P(C \cap M \cap S') = P(C) \times P(M|C) \times P(S'|M \cap C)$

$\qquad\qquad\qquad = \dfrac{1}{4} \times \dfrac{3}{5} \times \dfrac{1}{5} = \dfrac{3}{100}$ ⑤

∴ The required probability $= 1 - P(C \cap M | S')$ ⑤

$\qquad\qquad\qquad = 1 - \dfrac{\frac{3}{100}}{1 - \frac{87}{160}}$ ⑤ $= 1 - \dfrac{3 \times 160}{100 \times 73}$

$\qquad\qquad\qquad = \dfrac{682}{730}$ ⑤

☐20

(b) $u_i = \dfrac{x_i - A}{10}$

Class Mark	f_i	u_i	$f_i u_i$	$f_i u_i^2$
24.5	1	−3	−3	9
34.5	9	−2	−18	36
44.5	35	−1	−35	35
54.5	40	0	0	0
64.5	12	1	12	12
74.5	3	2	6	12
			−38	104
			⑤	⑤

(i) $A = 54.5$

$\overline{x} = A + C \dfrac{\sum\limits_{i=1}^{n} f_i u_i}{\sum\limits_{i=1}^{n} f_i} = 54.5 + 10\left(\dfrac{-38}{100}\right)$ ⑤

$\qquad = 54.5 - 3.8$

$\qquad = 50.7$ ⑤

Modal class 49.5 − 59.5

$M_0 = 49.5 + \left(\dfrac{40 - 35}{(40-35)+(40-12)}\right) \times 10$ ⑤

$\qquad = 49.5 + \dfrac{5}{33} \times 10$

$\qquad = 51.02$ ⑤

S.D $= \sigma = C\sqrt{\dfrac{\sum f_i u_i^2}{\sum f_i} - \left(\dfrac{f_i u_i}{\sum f_i}\right)^2} = 10\sqrt{\dfrac{104}{100} - \left(\dfrac{-38}{100}\right)^2}$ ⑤

$\qquad\qquad = 10\sqrt{(1.04 - 0.38^2)}$

$\qquad\qquad = 9.46$ ⑤

☐40

(ii) The actual mark corresponding to the mark x_i is $(x_i - 3)$

The actual mean $= \bar{x} - 3 = 50.7 - 3.00$

$= 47.7$ ⑤

The actual mode $M'_0 = m_0 - 3 = 48.02$

Since the variance does not change, the standard deviation also remains the same. ⑤

The standard deviation $= 9.46$ ⑤

$\boxed{15}$

(iii) $\bar{x} = \dfrac{100 \times 47.7 \times 50 \times 55}{150} = 50.13$ ⑤

$\sigma^2 = \dfrac{1}{n_1 + n_2}\left\{ n_1\sigma_1^2 + n_2\sigma_2^2 + \dfrac{n_1 n_2}{n_1 + n_2}(\bar{x}_1 - \bar{x}_2)^2 \right\}$ ⑤

$= \dfrac{1}{150}\left\{ 100 \times 9.46^2 + 50 \times 2.5^2 + \dfrac{50 \times 100}{150}(7.3)^2 \right\}$ ⑤

$= 73.59$

$\sigma = \sqrt{73.59} = 8.58$ ⑤

$\boxed{20}$