G.C.E.(A.L) Support Seminar - 2015

Combined Mathematics I | **Three Hours**

Part A

Answer **all** the questions in the given space.

1. By using the Principle of Mathematical Induction, prove that

$$\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^n = \begin{pmatrix} 1+2n & -4n \\ n & 1-2n \end{pmatrix}$$ for all positive integers n.

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2. By using the knowledge on binomial expansion, simplify $\left(\sqrt{5}+\sqrt{3}\right)^4 + \left(\sqrt{5}-\sqrt{3}\right)^4$.

Hence, find the integer n such that $n < \left(\sqrt{5}+\sqrt{3}\right)^4 < n+1$.

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3. Express $(3-2i)(7-5i)$ in the form of $x+iy$, where $x, y \in \mathbb{R}$, $i^2 = -1$.

Hence, deduce two factors of $11+29i$ and express 11^2+29^2 as a product of two positive integers.

4. Show that $\lim_{x \to \frac{\pi}{2}} \dfrac{(2x-\pi)\cos x}{2\cos^2 x - \left(\frac{\pi}{2}-x\right)^2 \sin x} = -2$.

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5. Find $\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(x+\sqrt{x^2+a^2}\right)$ and hence obtain $\int \frac{1}{\sqrt{9x^2+4}}\,\mathrm{d}x$. Here $a\in \mathbb{R}$.

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6. A curve is given by the equations $x = t(1-t)^2$, $y = t^2(1-t)$; where t is a real parameter. Taking that the gradient of the tangent to this curve at the point $\left(T(1-T)^2, T^2(1-T)\right)$ is $\frac{T(2-3T)}{(1-T)(1-3T)}$ where $T \neq 1, \frac{1}{3}$, show that the equation of the tangent drawn to the curve at the point $t = \frac{1}{2}$ is $4x + 4y - 1 = 0$.
Hence, show that the curve is completely located on one side of the above tangent.

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7. The equations of the sides AB, BC and CA of a triangle ABC are $3x+4y+5=0$, $3x-4y+1=0$ and $5y-2=0$ respectively. Find the equations of the bisectors of $A\hat{B}C$ and distinguish the interior angle bisector and the exterior angle bisector..

8. Find the values of a and b such that the equation $ax^2+2y^2+bxy+x+4y+2c=0$ represents a circle and show that $c<\frac{17}{16}$; where a, b and c are constants.
If the circle $(x+p)^2+y^2=p^2$ bisects the above circle for the positive integral value of c, find the value of the constant p.

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9. Find r which satisfies the condition that the two circles $S_1 : x^2 + y^2 - 6x + 8y + 9 = 0$ and $S_2 : x^2 + y^2 - r^2 = 0$ touch each other.
Obtain the coordinates of the point of contact of the two circles when they touch each other internally.

10. The lengths of the sides AB, BC, CA of a triangle ABC are 4 m, 5 m, 6 m respectively. Show that $\cos C = \frac{3}{4}$ and $\frac{\sin B}{\sin C} = \frac{3}{2}$, and hence deduce that $\hat{B} = 2\hat{C}$.

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Part B

Answer **five** questions only.

11. (a) Let α and β be the roots of the quadratic equation $x^2 + bx + c = 0$, where b and c are real numbers. Find the quadratic equation whose roots are $p = \alpha + \beta^2$ and $q = \beta + \alpha^2$.
Show that, when α and β are imaginary, p and q are real if and only if $b = -1$ and also show that $p = q = 1 - c$ in this instance.

(b) Show that $0 \le \dfrac{(x+2)^2}{x^2 + x + 1} \le 4$ for all real x.
Hence, sketch the graph of $y = \dfrac{(x+2)^2}{x^2 + x + 1}$.

(c) Find the values that k can take for $x^2 + kx + 1$ to be a factor of $x^4 - 12x^2 + 8x + 3$, and hence solve the equation $x^4 - 12x^2 + 8x + 3 = 0$.

12. (a) Show that the number of ways that Rs. 18 could be divided among five children in integral multiples of a rupee such that each child gets at least three rupees, is 35.

(b) Sketch the graphs of $y = |x - a|$ and $y = b|x - 1|$ on the same diagram. Here $a > b > 0$ and $b \neq 1$.
Hence, find the values of a and b, if the set of real values of x that satisfies the inequality $b|x - 1| > |x - a|$ is $\{x : 3 < x < 7\}$.

(c) Let $u_r \equiv \dfrac{3r+1}{(r+1)(r+2)(r+3)}$ for $r \in \mathbb{Z}^+$.

Find $f(r)$ and constants λ, μ such that $u_r \equiv \lambda\big(f(r) - f(r+1)\big) + \mu\big(f(r+1) - f(r+2)\big)$ for $r \in \mathbb{Z}^+$.

Show that $\displaystyle\sum_{r=1}^{n} u_r = \frac{5}{6} - \frac{3n+5}{(n+2)(n+3)}$ for $n \in \mathbb{Z}^+$.

Also show that the infinite series $\displaystyle\sum_{r=1}^{\infty} u_r$ is convergent and deduce that

$$\frac{1}{6} \le \frac{5}{6} - \frac{3n+5}{(n+2)(n+3)} < \frac{5}{6}.$$

13. (a) Let $\mathbf{P} = \begin{pmatrix} -5 & 3 \\ 6 & -2 \end{pmatrix}$. Find the two distinct real values of λ such that $\det(\mathbf{P} - \lambda\mathbf{I}) = 0$. Here $\mathbf{I}$ is the unit matrix of order 2×2.

For each value of λ, find the column matrix $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ which satisfies $\mathbf{PX} = \lambda\mathbf{X}$.

(b) The complex numbers z_1, z_2, z_3 and z_4 are respectively represented by the vertices of a quadrilateral $A_1 A_2 A_3 A_4$ on an Argand diagram.
Interpret the modulus and the amplitude of the complex number $\dfrac{z_1 - z_3}{z_2 - z_4}$ geometrically.

Find the geometrical requirement for $\dfrac{z_1 - z_3}{z_2 - z_4}$ to be purely imaginary.

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The roots of the equations $z^2-2z+2=0$ and $z^2-2az+b=0$ are represented on an Argand diagram by the distinct points A, B and C, D respectively. Here $a, b \in \mathbb{R}$ and $a^2<b$.
Show that,

(i) if $C\hat{O}D=\frac{\pi}{2}$, then $2a^2=b$.
(ii) if the points A, B, C, D are equidistant from O, then $b=2$.
(iii) if the points A, B, C, D are the vertices of a square with centre O, then $a=-1$ and $b=2$.
Here O is the origin.

(c) The variable complex number z is represented on an Argand diagram by the point P. If $\arg[(z+i)i]=\frac{2\pi}{3}$, find the locus of the point P.
Find also the minimum value of $|z|$ and the complex number represented by the point corresponding to the minimum value of $|z|$.

14. (a) Let $f(x)=\frac{3-4x}{x^2+1}$, where $x \in \mathbb{R}$.
By using the knowledge on the first derivative, show that the function f has two turning points, and sketch the graph of $y=f(x)$.
Using your graph, sketch the graph of $y=|f|(x)$ on another xy - plane.
Hence, show that the equation $|3-4x|e^x-x^2-1=0$ has at least three real roots.

(b) In a triangle ABC, $AB=AC$. Its perimeter is $2s$ where s is a constant. Find in terms of s, the length of AB such that the volume of the solid generated by rotating the triangle about BC is maximum.

15. (a) Let $f(x)=\frac{x^2+3x+5}{(x-1)(x+2)}$. Find the constants A, B, C such that $f(x)=A+\frac{B}{x-1}+\frac{C}{x+2}$.
Hence, evaluate $\int_0^2 f(x)\,\mathrm{d}x$.

(b) By using integration by parts, find $\int e^{2x}\sin 3x\,\mathrm{d}x$.

(c) Let $\mathrm{I}=\int_0^1 \frac{1}{x+\sqrt{1-x^2}}\,\mathrm{d}x$. Using the substitution $x=\sin\theta$,
show that $\mathrm{I}=\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta+\cos\theta}\,\mathrm{d}\theta$.

Using another suitable substitution show also that $\mathrm{I}=\int_0^{\frac{\pi}{2}} \frac{\sin\theta}{\sin\theta+\cos\theta}\,\mathrm{d}\theta$.
Hence show that $\mathrm{I}=\frac{\pi}{4}$.

(d) Find the area of the enclosed region given by $\{(x, y): x^2 \le y \le |x|\}$.

16. (a) In a triangle ABC, the equations of the sides AB, BC and CA are $x-2y+2=0$, $x-y-1=0$ and $2x-y-1=0$ respectively. The line which passes through A perpendicular to BC and the line which passes through B parallel to AC meet at D. Find the equations of the lines AD and BD. Show that $ABDC$ is a rhombus.

(b) Express the requirement for the circle with centre C_1, radius r_1 and the circle with centre C_2, radius r_2 to intersect each other.
The circles $S_1: x^2+y^2+6x+2fy=0$ and $S_2: x^2+y^2-2y-3=0$ intersect each other orthogonally. Show that $f=\frac{3}{2}$.
Show that any circle that passes through the intersection points of the circles $S_1=0$ and $S_2=0$ is given by $S_1+\lambda S_2=0$; where λ is a parameter.
Hence, obtain the equation of
(i) the circle that passes through the intersection points of the circles $S_1=0$ and $S_2=0$ and the point $(-2, 2)$.
(ii) the smallest circle that passes through the intersection points of the circles $S_1=0$ and $S_2=0$.

17. (a) In a triangle ABC, $\cos A+\cos B+\cos C=\frac{3}{2}$. Show that $\cos\frac{B-C}{2}=\frac{\left[1-2\sin\frac{A}{2}\right]^2}{4\sin\frac{A}{2}}+1$ and hence, deduce that the triangle ABC is equilateral.

(b) Express $f(\theta)\equiv 3\cos^2\theta+10\sin\theta\cos\theta+27\sin^2\theta$ in the form $a+b\cos(2\theta+\alpha)$; where a, b are constants and α is an acute angle independent of θ.
Sketch the graph of $y=f(\theta)$ in the interval $[0, \pi]$.
Using the graph, determine for which intervals of values of k, the equation $f(\theta)-k=0$ has
(i) only one solution
(ii) two solutions
(iii) three solutions
(iv) no solutions.

(c) Solve the equation $\sin^{-1}\sqrt{\frac{2}{3}}-\sin^{-1}x=\frac{\pi}{2}$.

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G.C.E.(A.L) Support Seminar - 2015

Combined Mathematics II | **Three Hours**

* In this question paper g denotes the acceleration due to gravity.

Part A

Answer **all** the questions in the given space.

1. At a certain instant, a boat travelling with uniform velocity is located 24 km to the East of a certain ship travelling with uniform velocity. Exactly two hours later the boat is located 7 km South of the ship.
   (i) Find the velocity of the boat relative to the ship and calculate the shortest distance between the boat and the ship.
   (ii) If the ship travels with a velocity of $13\,\text{km h}^{-1}$ at an angle $\tan^{-1}\left(\frac{12}{5}\right)$ East of North, find the actual velocity of the boat.

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2. $AB = 4a$ and $BC = 3a$ in a rectangle $ABCD$. Four particles of mass m each are kept at rest at the four vertices of the rectangle and are connected by four light inextensible strings AB, BC, CD and DA. All the strings are taut. If an impulse I is given to particle A in the direction of CA, find the initial velocity of the motion of each particle.

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3. A child can ride a bicycle along a horizontal road at a maximum velocity of $3\,\mathrm{m\,s^{-1}}$. He can cycle upwards on the same bicycle along a similar road inclined at an angle 30° to the horizontal at a maximum velocity of $2\,\mathrm{m\,s^{-1}}$. What is the acceleration of the bicycle at the instant when the child is cycling with velocity $4\,\mathrm{m\,s^{-1}}$ down the same inclined road?
Consider that the mass of the child and the bicycle together is 95 kg, that the child cycles with a constant power, that the resistance to the motion is proportional to the square of the velocity and that $g = 10\,\mathrm{m\,s^{-2}}$.

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4. Two particles P and Q which start their motions simultaneously from the points A and B with position vectors $-10\,\mathbf{i} + 6\,\mathbf{j}$ and $2\,\mathbf{i} + 3\,\mathbf{j}$ respectively, travel with uniform velocities $\mathbf{i} + \mathbf{j}$ and $\mathbf{v}$ respectively. The velocity $\mathbf{v}$ is parallel to $-2\,\mathbf{i} + \mathbf{j}$. If the particles P and Q collide, find the time to the collision and show that $|\mathbf{v}| = \dfrac{5\sqrt{5}}{2}$. Here $\mathbf{i}$ and $\mathbf{j}$ are unit vectors which are perpendicular to each other.

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5. If $\overrightarrow{OA} = \mathbf{a} + 2\mathbf{b}$, $\overrightarrow{OB} = 3\mathbf{a} - \mathbf{b}$ and $OA \perp OB$, show that $\mathbf{a} \cdot \mathbf{b} = \frac{2}{5}|\mathbf{b}|^2 - \frac{3}{5}|\mathbf{a}|^2$.

If $|\mathbf{a}| = 2$ and $|\mathbf{b}| = 1$, find the angle between **a** and **b**.

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6. State the necessary condition for a system consisting of only three coplanar forces which are not parallel to each other to be in equilibrium.
A uniform cylinder lies on a rough horizontal plane with its axis horizontal. A heavy rod in a vertical plane through the centre of gravity of the cylinder, is in equilibrium touching the curved surface of the cylinder, with one end of the rod in contact with the horizontal plane. The contact between the cylinder and the rod is rough. If the rod is inclined at an angle θ to the horizontal and if the angle of friction between the rod and the cylinder is λ, show that $\lambda \geq \frac{\theta}{2}$.

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7. The probabilities of three children A, B and C being able to independently solve a problem correctly are $\frac{1}{6}$, $\frac{1}{2}$ and $\frac{1}{3}$ respectively. Find the probability of exactly two of these children being able to independently solve this problem correctly.

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8. The probability of a pink flowered plant growing from a certain type of seed is $\frac{1}{6}$. Find the minimum number of such seeds that need to be planted for the probability of at least one pink flowered plant growing to exceed 0.98. (Take $\frac{\ln(0.02)}{\ln\left(\frac{5}{6}\right)} = 21.46$)

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9. Find the mean and standard deviation of the values that x takes when each of the observations 2004, 2008, 2000, 2008, 1996, 1992, 2000, 2008, 2008 and 2000 is expressed in the form $2000 - 4x$.

Hence, find the mean and standard deviation of the given observations.

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10. The mean of the marks of twenty children for a question paper in combined mathematics is 40. The mean of the six least marks is 25. The highest six marks are 70, 71, 72, 74, 75 and 78.
Find, (i) the mean of the remaining eight marks.

(ii) the third quartile of all the marks.

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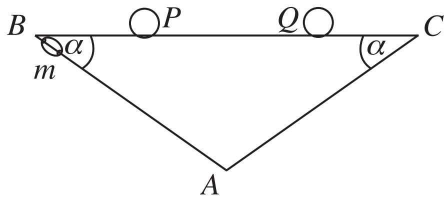
Part B

Answer **five** questions only.

11. (a) An elevator starts its motion from rest at time $t = 0$ and moves vertically upwards with uniform acceleration a. A man who is in the elevator releases a particle P from rest under gravity at time $t = t_0$. At the instant when the particle P reaches its maximum height, a second particle Q is released from rest under gravity. Sketch the velocity time graphs for the motions of the elevator and the two particles P and Q on the same diagram.
Hence, show that at the instant when Q comes to instantaneous rest, the velocity of P is $at_0\left(\frac{a}{g}+1\right)$.

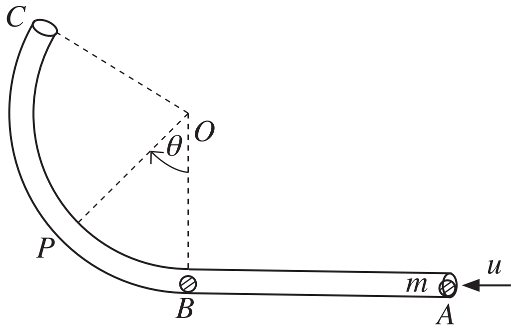
(b) A and B are two points that lie on two straight roads which meet at O. $OA = a$ km, $OB = b$ km and $A\hat{O}B = \alpha$. Two vehicles X and Y travelling towards O, pass the points A and B respectively with speeds of 20 km h^{-1} and 40 km h^{-1} and accelerations of 5 km h^{-2} and 10 km h^{-2} respectively. Show by considering velocity triangles and acceleration triangles, that the path of Y with respect to X is a straight line.
Show also that if $a = b = 10$ and $\alpha = 60°$, then the shortest distance between X and Y is 5 km.

(c) As shown in the figure, a uniform frame ABC in the shape of an isosceles triangle and of mass M, has been placed in a vertical plane such that its side BC can freely slide along two smooth fixed rings P and Q on the same horizontal level. $\hat{B} = \hat{C} = \alpha$. A smooth bead of mass m which is able to slide freely along BA is kept at rest at B and the system is released from rest.



Show that the magnitude of the acceleration of the frame in the subsequent motion is $\frac{mg\sin 2\alpha}{2(M+m\sin^2\alpha)}$ and that the reaction on the bead is $\frac{Mmg\cos\alpha}{M+m\sin^2\alpha}$.

12. (a) A smooth straight tube AB and a portion BC of a smooth circular tube of radius a, with an angle of $\frac{2\pi}{3}$ subtended at the centre, both of equal cross section, have been connected together as shown in the figure. This composite tube is fixed in a vertical plane such that AB lies on a horizontal plane. A particle of mass m is placed at point A and another particle of mass $2m$ is placed at the point B. The particle of mass m is projected into the tube with horizontal velocity u. The two particles collide and coalesce.



(i) Find the velocity of the composite particle of mass $3m$ after the collision.

(ii) Find the velocity of the composite particle at the point P, and the reaction between the composite particle and the tube at this point, where OP is inclined at an angle θ to the downward vertical.

If the composite particle moves through the tube and leaves it at C,

(iii) then show that $u > 3\sqrt{3ag}$.

(iv) If $AB = \sqrt{3}\,a$ and if the particle of mass $3m$ which leaves the tube at C falls at A, show that $u = \frac{3}{2}\sqrt{21ag}$.

(b) A smooth sphere of mass m, moving with speed u on a smooth horizontal table, collides directly with a sphere of the same radius, but of mass M which is at rest. If half the kinetic energy is lost in the impact, show that $e < \frac{1}{\sqrt{2}}$; where e is the coefficient of restitution between the two spheres.

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13. Two particles A and B of mass M and m respectively, which are attached to the two ends of a light elastic string of natural length a and modulus of elasticity $2\ mg$, are at rest on a rough horizontal table such that the string is taut. The coefficient of friction between each particle and the table is $\frac{1}{2}$. Show that when particle B is given a velocity $\sqrt{ga}$ along the table away from A, its motion when the extension of the string is x, is given by the equation $\ddot{x} = -\frac{2g}{a}\left(x + \frac{a}{4}\right)$, by assuming that particle A is at rest throughout.

By assuming that this equation has a solution of the form $x + \frac{a}{4} = \alpha\cos\omega t + \beta\sin\omega t$, find α, β and ω.

Hence show that the maximum extension of the string is $\frac{a}{2}$.
Show also that $M \geq 2m$.

Show that the return motion of the particle B is given by the equation $\ddot{y} = -\frac{2g}{a}\left(y - \frac{a}{4}\right)$. Here y is the extension of the string.

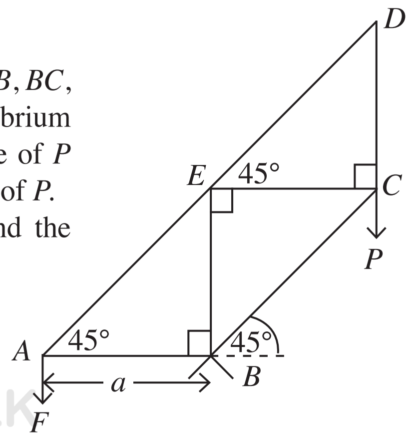
By assuming that the solution of this equation is $y = \frac{a}{4}\left[1 + \cos\sqrt{\frac{2g}{a}}t\right]$, show that particle B comes to a definite rest at the initial point after time $\left[\pi + \cos^{-1}\left(\frac{1}{3}\right)\right]\sqrt{\frac{a}{2g}}$.

14. (a) The position vectors of the two points A and B with respect to point O are **a** and **b** respectively. The point E on OA is such that $OE : EA = 3 : 4$ and the point D on OB is such that $OD : DB = 5:2$. If G is the point of intersection of the straight lines AD and BE, show that $\overrightarrow{OG} = \mathbf{b} + \lambda\left(\frac{3}{7}\mathbf{a} - \mathbf{b}\right)$. Here λ is a constant.
Obtain another such expression for $\overrightarrow{OG}$ and find the position vector of G in terms of **a** and **b**.

(b) $AB = 4a$ and $BC = 3a$ in the rectangle $ABCD$. The forces $3\mathbf{P}$, $4\mathbf{P}$, $2\mathbf{P}$, $\mathbf{P}$, $\lambda\mathbf{P}$ and $\mu\mathbf{P}$ act along AB, BC, CD, DA, AC and BD respectively in the direction indicated by the order of the letters. Find the values of λ and μ for which this system of forces is equivalent to
(i) a couple
(ii) a force through B parallel to AC.
Show also that there are no values of λ and μ for which the system is in equilibrium.

15. (a) A frame $ABCDE$ in the shape of a regular pentagon has been made with five identical uniform rods, each of length $2a$ and weight w, freely jointed at their ends. The frame has been kept in a vertical plane with the side CD of the pentagon in contact with a horizontal plane. The shape of the pentagon is maintained by a string joining the midpoints of the rods BC and DE. Show that the reaction at the joint A is $\frac{w}{2}\cot\frac{\pi}{5}$.
Find also the tension in the string.

(b) The framework in the figure consists of seven light rods AB, BC, CD, DE, EA, BE and EC. The framework is kept in equilibrium with B on a smooth support, by applying a vertical force of P Newtons at C and a vertical force F at A. Find F in terms of P. By using Bow's notation draw a stress diagram and find the stress in each rod and the reaction on the support at B.



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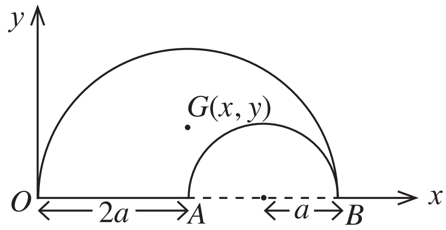
16. Show using integration that the centre of mass of a thin uniform semi-circular wire of radius a is at a distance $\frac{2a}{\pi}$ from the centre O.



The frame in the figure has been made by bending a thin uniform wire. If the centre of mass of the frame is $G(x, y)$, find x and y.
When the frame is freely suspended from O, if the angle of inclination of OA to the vertical is θ, show that $\tan\theta = \frac{10}{7\pi + 2}$.

While the frame is suspended from O, the edge OA is kept vertical by applying a horizontal force P at B, in the plane of the frame. If the weight of the frame is w, find the value of P in terms of w.

If instead of suspending it from O, the frame is kept in equilibrium in a vertical plane with its curved edge in contact with a horizontal plane, find the angle of inclination of OA to the horizontal.

17. (a) Let A and B be two random events. Define the conditional probability $P(A|B)$ when $P(B) > 0$.

Show that $P(A_1 \cap A_2 \cap A_3) = P(A_1).\ P(A_2|A_1).\ P(A_3|A_1 \cap A_2)$ for three random events A_1, A_2, A_3.

In a sports club, $\frac{3}{4}$ of the members are adults while the rest are children. Furthermore, $\frac{3}{4}$ of the adults and $\frac{3}{5}$ of the children are males. Exactly half of the male adults, $\frac{1}{3}$ of the female adults, $\frac{4}{5}$ of the male children and $\frac{4}{5}$ of the female children use the swimming pool. Find the probability of a person selected at random from the members of this club being,

(i) a person who uses the swimming pool.
(ii) a male, given that the person uses the swimming pool.
(iii) a female or an adult, given that the person does not use the swimming pool.

(b) The following table gives the class mark and corresponding frequency of a grouped frequency distribution of the marks obtained by 100 students who sat a certain examination.

Class Mark	**Frequency**
24.5	1
34.5	9
44.5	35
54.5	40
64.5	12
74.5	3

(i) By using a suitable coding method, show that the mean of this distribution is 50.7, the mode is 51.02 and the standard deviation is 9.46.
(ii) It was later found that each mark entered in the above distribution was 3 more than the actual mark. Find the mean, mode and standard deviation of the actual distribution.
(iii) If the actual mean and actual standard deviation of another 50 students are 55 and 2.5 respectively, find the mean and standard deviation of the combined marks.

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