G.C.E. (A.L.) Support Seminar - 2016
Combined Mathematics - Paper I
Answer Guide

Part A

1. Let $f(n) = 4^n + 15n - 1;\ n \in \mathbb{Z}^+$

When $n = 1,\ f(1) = 4 + 15 - 1 = 18 = 9 \times 2$

$\therefore\ f(1)$ is divisible by 9.

$\therefore$ The statement is true when $n = 1$. (5)

Let us assume that the given expression is divisible by 9 when $n = p,\ p \in \mathbb{Z}^+$

That is, $f(p) = 4^p + 15p - 1 = 9k;\ k \in \mathbb{Z}^+$. (5)

$$f(p+1) = 4^{p+1} + 15(p+1) - 1$$

$$= 4.\,4^p + 15p + 15 - 1$$

$$= 4\,[9k - 15p + 1] + 15p + 15 - 1 \quad (5)$$

$$= 4 \times 9k - 45p + 18$$

$$= 9\,[4k - 5p + 2]$$

$$= 9\lambda;\ \lambda = 4k - 5p + 2 \in \mathbb{Z}^+$$

$\therefore\ f(p+1)$ is divisible by 9.

$\therefore$ The statement is true when $n = p + 1$ (5)

$\therefore$ By the Principle of Mathematical Induction, the given expression is divisible by 9 for all positive integers n. (5)

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2.

$$\left(\sqrt{2} + 7^{\frac{1}{5}}\right)^{10} = \sum_{r=0}^{10} {}^{10}C_r \left(2^{\frac{1}{2}}\right)^{10-r} \left(7^{\frac{1}{5}}\right)^r$$

$T_r = {}^{10}C_{r-1} \left(2\right)^{\frac{11-r}{2}} \left(7\right)^{\frac{r-1}{5}}$; here $1 \le r \le 11$. (5)

Since 2 and 7 are primes, for a term to be rational $11 - r = 2p$ and $r - 1 = 5q$;

where $p, q \in \mathbb{Z}^+$. (5)

That is $r \in \{1, 3, 5, 7, 9, 11\} \cap \{1, 6, 11\}$ (5)

$\therefore\ r = 1$ or 11. (5)

$\therefore$ The sum of the rational terms $= {}^{10}C_0\, 2^5 + {}^{10}C_{10}\, 7^2 = 32 + 49 = 81$

(5)

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[see page 2

3. The number of ways in which a group of 5 can be formed without restriction $= {}^{14}C_5$

$= 2002$ (5)

The number of groups with 5 boys $= {}^{8}C_5$

$= 56$ (5)

The number of groups with 5 girls $= {}^{6}C_5$

$= 6$ (5)

$\therefore$ The number of ways in which a group of 5 can be formed such that both sexes are represented in the group $= {}^{14}C_5 - \left({}^{8}C_5 + {}^{6}C_5\right)$

$= 2002 - 56 - 6$ (5)

$= 1940$ (5)

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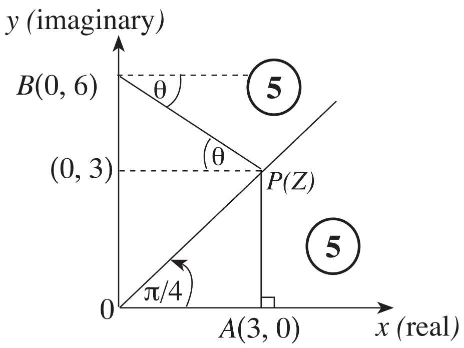
4.



As indicated in the figure, the point which corresponds to the complex number $Z = Z_0$ such that $\text{Arg } Z = \frac{\pi}{4}$ and $\text{Arg }(Z - 3) = \frac{\pi}{2}$ is P. (5) According to the figure $\theta = \frac{\pi}{4}$ (5)

$\therefore \text{ Arg }(Z_0 - 6i) = \frac{7\pi}{4}$ (5)

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5.

$$\lim_{x \to 0} \frac{(1 + kx)^2 - (1 - kx)^2}{\sqrt{1 + k^2x} - \sqrt{1 - k^2x}}$$

$$= \lim_{x \to 0} \frac{1 + 2kx + k^2x^2 - 1 + 2kx - k^2x^2}{(1 + k^2x) - (1 - k^2x)} \times \left(\sqrt{1 + k^2x} + \sqrt{1 - k^2x}\right) \quad (10)$$

$$= \lim_{x \to 0} \frac{4kx}{2k^2x} \left(\sqrt{1 + k^2x} + \sqrt{1 - k^2x}\right) \quad ; k, x \neq 0$$

$$= \frac{2}{k} \lim_{x \to 0} \left(\sqrt{1 + k^2x} + \sqrt{1 - k^2x}\right) = \left(\frac{2}{k}\right) \times 2 = \frac{4}{k} \quad (5)$$

$$\frac{4}{k} = 1 \quad (5)$$

$$\therefore k = 4 \quad (5)$$

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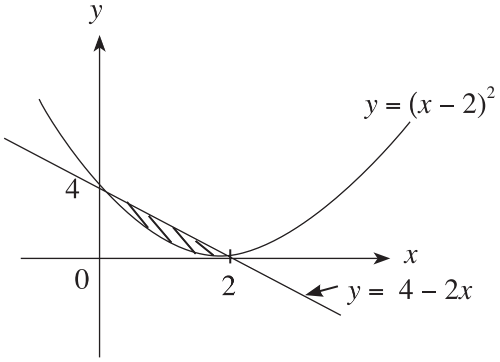
[*see page* 3

6.



(5)

$$\text{Area} = \int_0^2 \left\{ (4-2x) - (x-2)^2 \right\} dx \quad (5)$$

$$= \int_0^2 (4-2x)dx - \int_0^2 (x-2)^2\, dx$$

$$= \left[4x - \frac{2x^2}{2}\right]_0^2 - \left[\frac{(x-2)^3}{3}\right]_0^2$$

(5) (5)

$$= (8-4) - \left[0 + \frac{8}{3}\right]$$

$$= 4 - \frac{8}{3}$$

$$= \frac{4}{3} \quad (5)$$

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7. Differentiating with respect to t,

$$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = 3at^2 - 2t \quad (5)$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{dy}{dx} = (3at^2 - 2t) \cdot \frac{1}{2t} = \frac{3at-2}{2} \;;\; t \neq 0 \quad (5)$$

$$\left(\frac{dy}{dx}\right)_{t=1} = \frac{3a-2}{2} \quad , \quad \left(\frac{dy}{dx}\right)_{t=-1} = \frac{-3a-2}{2} \quad (5)$$

Since the tangents are perpendicular to each other,

$$\left(\frac{3a-2}{2}\right)\left(\frac{-3a-2}{2}\right) = -1 \quad (5)$$

$$9a^2 - 4 = 4 \quad \Rightarrow \quad a^2 = 8/9$$

Since $a > 0$, $a = \frac{2\sqrt{2}}{3}$ (5)

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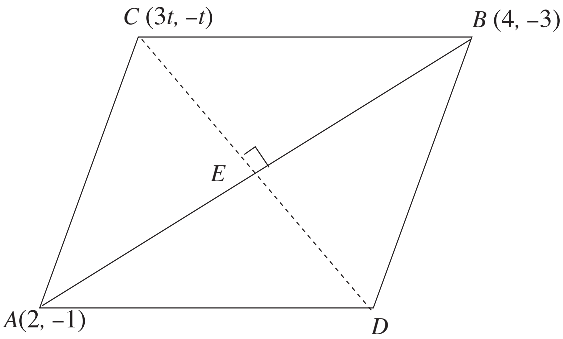
[*see page* 4

8.



$E = (3, -2)$

Since AB is perpendicular to CE,

$m_{AB} \times m_{CE} = -1.$

$\therefore -1 \times \left(\dfrac{-2+t}{3-3t}\right) = -1$ (5)

$\Rightarrow t = \dfrac{5}{4}$ (5)

$\therefore C = \left(\dfrac{15}{4}, -\dfrac{5}{4}\right)$ (5)

Let $D = (\overline{x}, \overline{y})$. Then, (5)

$\overline{x} = 2 \times 3 - 3t = 6 - 3 \times \dfrac{5}{4} = \dfrac{9}{4}$

$\overline{y} = 2 \times -2 + t = -4 + \dfrac{5}{4} = -\dfrac{11}{4}$

$\therefore D = \left(\dfrac{9}{4}, -\dfrac{11}{4}\right)$ (5)

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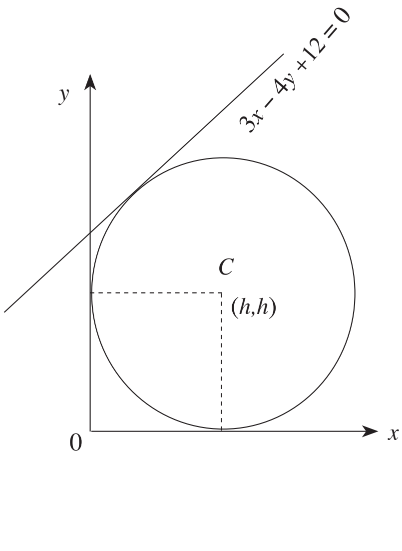
9.



[*see page* 5

Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be a required circle.

Since the circle touches the x and y axes, $C = (h,h)$ (5)

Furthermore, since the line $3x - 4y + 12 = 0$ touches the circle,

$$\frac{|3h - 4h + 12|}{\sqrt{3^2 + 4^2}} = |h|$$ (5)

$$|-h + 12| = 5|h|$$ (5)

$$\Leftrightarrow (-h + 12) = \pm 5h$$

$\therefore$ $h = -3$ or $h = 2$. (5)

$\therefore$ the equations of the circles are

$$(x-2)^2 + (y-2)^2 = 2^2$$

$$(x+3)^2 + (y+3)^2 = 3^2$$ (5)

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10. $\cot\alpha - \tan\alpha$

$$= \frac{1}{\tan\alpha} - \tan\alpha$$

$$= \frac{1 - \tan^2\alpha}{\tan\alpha}$$

$$= \frac{2(1 - \tan^2\alpha)}{2\tan\alpha} = \frac{2}{\tan 2\alpha}$$

$$= 2\cot 2\alpha$$ (5)

$\cot\alpha - \tan\alpha = 2\cot 2\alpha$ ——— (1)

$\therefore$ $\cot 2\alpha - \tan 2\alpha = 2\cot 4\alpha$ ——— (2) (5)

$\cot 4\alpha - \tan 4\alpha = 2\cot 8\alpha$ ——— (3) (5)

From (1) + 2 × (2) + 4 × (3)

$\cot\alpha - \tan\alpha - 2\tan 2\alpha - 4\tan 4\alpha = 8\cot 8\alpha$ (10)

$\cot\alpha = \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha$

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[*see page* 6

Part B

11. (a) $ax^2 + bx + c = 0$

$a\left[x^2 + \frac{bx}{a} + \frac{c}{a}\right] = 0$

$a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right] = 0$

$a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{(b^2 - 4ac)}{4a^2}\right] = 0$ (10)

The condition to have coincident roots is $b^2 - 4ac = 0$. (10)

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$\frac{a}{x+c} + \frac{b}{x-c} = \frac{k}{2x}$

$\frac{a(x-c) + b(x+c)}{x^2 - c^2} = \frac{k}{2x}$

$x^2[k - 2a - 2b] - 2(bc - ac)x - kc^2 = 0$ (10)

For coincident roots,

$4(bc - ac)^2 - 4(k - 2a - 2b)(-kc^2) = 0$ (10)

That is, $k^2 - 2(a+b)k + (b-a)^2 = 0$ (5)

If k_1 and k_2 are the roots of the above equation, then

$k_1 + k_2 = 2(a+b)$ (5) $k_1 k_2 = (b-a)^2$ (5)

$(k_1 - k_2)^2 = (k_1 + k_2)^2 - 4k_1k_2$ (10)

$= 4(a+b)^2 - 4(b-a)^2$

$= 16ab$

$\therefore |k_1 - k_2| = 4\sqrt{ab}$ (10)

55

(b) $f(x) = (\lambda + 1)x^2 + (6 - 3\lambda)x + (20 - 12\lambda)$

(i) $f(x)$ is linear when $\lambda = -1$. (5)

(ii) Let the roots be α and $-\alpha$. (5)

Then $\alpha + (-\alpha) = -\frac{(6 - 3\lambda)}{(\lambda + 1)}$ (5)

$\therefore 0 = 6 - 3\lambda$. Hence $\lambda = 2$. (5)

[see page 7

(iii) $f(x) = h - b(x - a)^2 = h - b(x^2 - 2ax + a^2) = -bx^2 + 2abx + (h - ba^2)$

$f(x) = (\lambda + 1)x^2 + (6 - 3\lambda)x + (20 - 12\lambda)$ (5)

$\therefore$ by comparing the coefficients, $-b = \lambda + 1 \Rightarrow b = -(\lambda + 1)$ ——— (1) (5)

$2ab = 6 - 3\lambda \Rightarrow a = -\dfrac{3(2-\lambda)}{2(\lambda+1)}$ ——— (2) (5)

$h - ba^2 = 20 - 12\lambda \Rightarrow h = 4(5 - 3\lambda) - \dfrac{9}{4}\dfrac{(2-\lambda)^2}{(\lambda+1)}$ ——— (3) (10)

Since the maximum value of $f(x)$ occurs at $x = 2$, we obtain that $a = 2$. (5)

(2) $\Rightarrow 4(\lambda + 1) = -(6 - 3\lambda) \Rightarrow 4\lambda + 4 = -6 + 3\lambda \Rightarrow \lambda = -10$ (5)

$h = 4(5 + 30) - \dfrac{9}{4}\dfrac{(2+10)^2}{(-10+1)}$ (10)

$\therefore$ the maximum value of $f(x) = 176$ (10)

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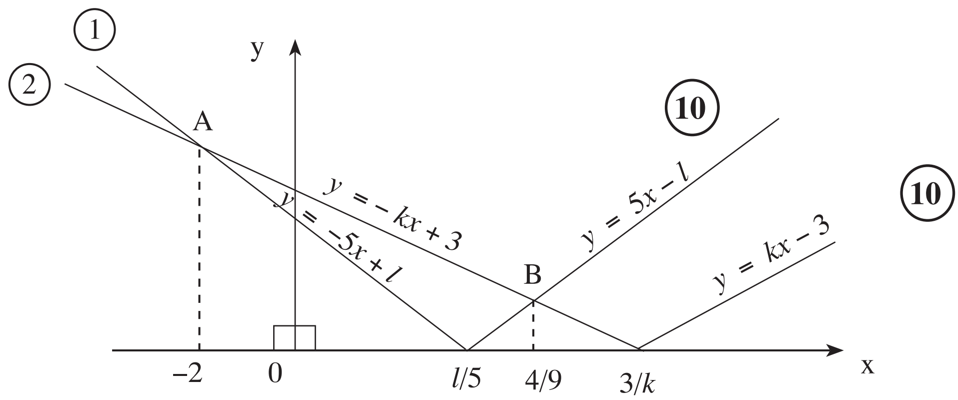
12. (a) Since the solution set of the inequality $|l - 5x| < |kx - 3|$ is $\{x \mid -2 < x < 4/9\}$, the two graphs are as illustrated below



(1) $y = |l - 5x|$

(2) $y = |kx - 3|$

For point A: $l + 10 = 2k + 3$ (5)

$l - 2k = -7$ ——— (i) (5)

For point B: $-l + 5 \cdot \dfrac{4}{9} = -k \cdot \dfrac{4}{9} + 3$ (5)

$-9l + 4k = 7$ ——— (ii) (5)

by (i) and (ii) $l = 1$, (5) $k = 4$ (5)

50

[*see page* 8

(b) $S_n = \dfrac{3n}{2n+1}$ (5)

$\lim\limits_{n\to\infty} S_n = \dfrac{3}{2}$ (5)

The limit is finite. (5)

Therefore, the series is convergent. (5)

$U_r = S_r - S_{r-1}$ (5)

$= \dfrac{3r}{2r+1} - \dfrac{3(r-1)}{2r-1}$ (5)

$U_r = \dfrac{3}{4r^2-1}$ (5)

Let $S_n^{'} = \sum\limits_{r=1}^{n} r^2 \dfrac{3}{4r^2-1}$ (5)

$= \sum\limits_{r=1}^{n} \dfrac{\frac{3}{4}(4r^2-1) + \frac{3}{4}}{(4r^2-1)}$ (5)

$= \sum\limits_{r=1}^{n} \dfrac{3}{4} + \dfrac{1}{4}\sum\limits_{r=1}^{n} \dfrac{3}{4r^2-1}$ (10)

$= \dfrac{3n}{4} + \dfrac{1}{4} S_n$ (5)

(5)

$= \dfrac{3n}{4} + \dfrac{1}{4}\,\dfrac{3n}{(2n+1)}$ (5)

$= \dfrac{3n}{4}\left\{1 + \dfrac{1}{2n+1}\right\}$ (5)

$= \dfrac{3n(n+1)}{2(2n+1)}$ (5)

Therefore $\lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} r^2 U_r = \lim\limits_{n\to\infty} \dfrac{3n}{4}\left\{1 + \dfrac{1}{2n+1}\right\}$ (5)

$= \infty$ (5)

The limit is not finite. (5)

$\therefore$ the series is not convergent. (5)

100

[*see page* 9

13. (a)

$$\det A = \begin{vmatrix} 3 & p \\ -2 & -3 \end{vmatrix} = -9 + 2p \quad (5)$$

A^{-1} exists only if $\det A \neq 0$.

That is, if $p \neq 9/2$ (5)

$$A^{-1} = \frac{1}{(2p-9)} \begin{bmatrix} -3 & -p \\ 2 & 3 \end{bmatrix} \quad (5)$$

$$A^{-1} = A$$

$$\frac{1}{(2p-9)} \begin{bmatrix} -3 & -p \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & p \\ -2 & -3 \end{bmatrix} \quad (5)$$

Comparing the corresponding elements,

$$-\frac{3}{2p-9} = 3 \qquad \frac{-p}{2p-9} = p \quad (5)$$

$$\frac{2}{2p-9} = -2, \qquad \frac{3}{2p-9} = -3 \quad (5)$$

(5) (5)

$\Rightarrow 2p - 9 = -1$ and $p[1 + 2p - 9] = 0$

Since $p \neq 0$ we have that $p = 4$ (5)

$$\text{Therefore, } A = \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$$

$$A^{-1} = A$$

$$\Rightarrow A A^{-1} = A \, . \, A = A^2 \quad (5)$$

$$\therefore I = A^2$$

$$\Rightarrow 0 = A^2 - I$$

$$\Rightarrow 0 = (A - I)(A + I) \, ; \, I^2 = I \quad (5)$$

This is of the form, $0 = BC$,

$$\text{where } B = A - I = \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(5)

$$= \begin{bmatrix} 2 & 4 \\ -2 & -4 \end{bmatrix} = 2\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

(5)

[*see page* 10

and

$C = A + I = \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (5)

$= \begin{bmatrix} 4 & 4 \\ -2 & -2 \end{bmatrix} = 2\begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$ (5)

75

(b) (i) Let $Z = x + iy$, where $x, y \in \mathbb{R}$ (5)

$Z\,\overline{Z} = (x + iy)(x - iy)$

$= x^2 + y^2$ (5)

$= \left(\sqrt{x^2 + y^2}\right)^2 = |Z|^2$

$\therefore\ Z\,\overline{Z} = |Z|^2$.

10

(ii) Let $Z_1 = x_1 + iy_1$ and $Z_2 = x_2 + iy_2$ where $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

$Z_1 Z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$= x_1x_2 + i\,x_1y_2 + i\,y_1x_2 + i^2y_1y_2$

$= (x_1x_2 - y_1y_2) + i\,(x_1y_2 + y_1x_2)$ (5)

$\therefore\ \overline{Z_1Z_2} = (x_1x_2 - y_1y_2) - i\,(x_1y_2 + y_1x_2)$

$= x_1(x_2 - i\,y_2) - iy_1(-i\,y_2 + x_2)$

$= (x_1 - iy_1)(x_2 - i\,y_2)$ (5)

10

$\overline{Z_1Z_2} = \overline{Z}_1\,\overline{Z}_2$

(iii) $\left|\dfrac{\overline{Z}_1 - 2\overline{Z}_2}{2 - Z_1\overline{Z}_2}\right| = 1$

$\Rightarrow\ \left|\overline{Z}_1 - 2\overline{Z}_2\right| = \left|2 - Z_1\overline{Z}_2\right|$ (5)

$\Rightarrow\ \left|\overline{Z}_1 - 2\overline{Z}_2\right|^2 = \left|2 - Z_1\overline{Z}_2\right|^2$ (5)

$\Rightarrow\ \left(\overline{Z}_1 - 2\overline{Z}_2\right)\left(\overline{\overline{Z}_1 - 2\overline{Z}_2}\right) = \left(2 - Z_1\overline{Z}_2\right)\left(\overline{2 - Z_1\overline{Z}_2}\right)$ (5)

$\Rightarrow\ \left(\overline{Z}_1 - 2\overline{Z}_2\right)\left(Z_1 - 2Z_2\right) = \left(2 - Z_1\overline{Z}_2\right)\left(2 - \overline{Z}_1Z_2\right)$ (5)

[see page 11

$$Z_1\overline{Z}_1 - 2\overline{Z}_1 Z_2 - 2\overline{Z}_2 Z_1 + 4 Z_2\overline{Z}_2 = 4 - 2\overline{Z}_1 Z_2 - 2 Z_1\overline{Z}_2 + Z_1\overline{Z}_1 Z_2\overline{Z}_2 \quad (5)$$

$$|Z_1|^2 + 4|Z_2|^2 = 4 + |Z_1|^2\,|Z_2|^2$$

$$|Z_1|^2 + 4|Z_2|^2 - |Z_1|^2 \cdot |Z_2|^2 - 4 = 0$$

$$|Z_1|^2\left(1 - |Z_2|^2\right) - 4\left(1 - |Z_2|^2\right) = 0$$

$$\left(1 - |Z_2|^2\right)\left(|Z_1|^2 - 4\right) = 0 \quad (5)$$

Since $|Z_2| \neq 1,\ |Z_1|^2 - 4 = 0$

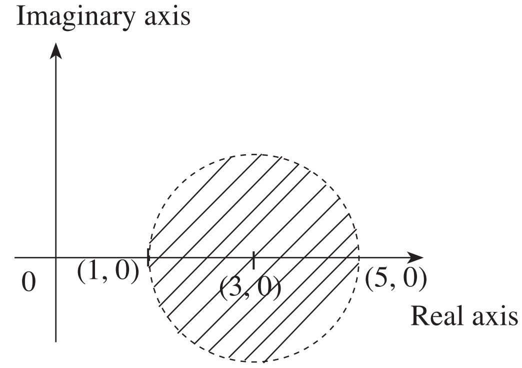
$\therefore\ |Z_1|^2 = 4$

Since $|Z_1| > 0,\ |Z_1| = 2$ (5)

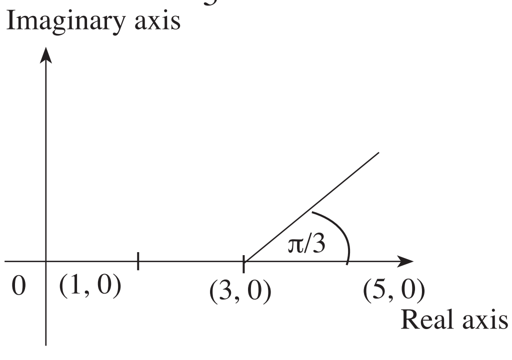
35

(c)

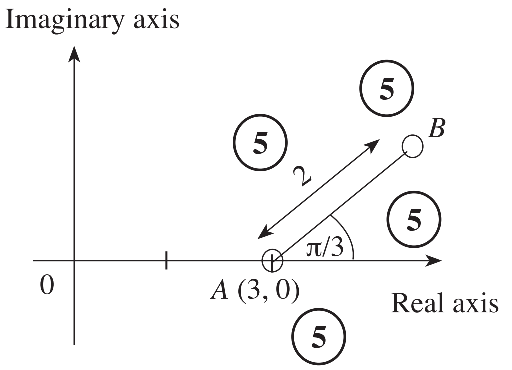
$|Z - 3| < 2$



$\text{Arg}\,(Z - 3) = \dfrac{\pi}{3}$



$\therefore$ The locus of the point *P* which represents the complex number *Z* such that $|Z - 3| < 2$ and $\text{Arg}\,(Z - 3) = \dfrac{\pi}{3}$;



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[*see page* 12

14. (a) $y = (\sin x)^x \qquad 0 \le x \le \frac{\pi}{2}$

$\ln y = x \ln|\sin x|$ (10)

$\frac{1}{y}\frac{dy}{dx} = \ln|\sin x| + x\cot x$ (10)

$\therefore \frac{dy}{dx} = \left[x\cot x + \ln(\sin x)\right](\sin x)^x$ (5)

25

(b) Volume of the tank $= \pi x^2 y + \frac{2}{3}\pi x^3$

$\therefore \pi x^2 y + \frac{2}{3}\pi x^3 = 45\pi$ (5)

$\therefore 45 = x^2(y + \frac{2}{3}x)$

$y = \frac{45}{x^2} - \frac{2}{3}x$ (5)

Surface area of the tank

$A = 2\pi x^2 + \pi x^2 + 2\pi xy$ (10)

$A = 3\pi x^2 + 2\pi xy$

$A = 3\pi x^2 + 2\pi x\left(\frac{45}{x^2} - \frac{2}{3}x\right)$

$A = 3\pi x^2 + \frac{90\pi}{x} - \frac{4\pi}{3}x^2$

$A = \frac{5\pi x^2}{3} + \frac{90\pi}{x}$ (5)

$\frac{dA}{dx} = \frac{10\pi x}{3} - \frac{90\pi}{x^2}$ (5)

$= \frac{10\pi(x^3 - 27)}{3x^2}$

$= \frac{10\pi}{3x^2}(x-3)(x^2 + 3x + 9)$ (5)

$\frac{dA}{dx} = 0$ when $x = 3$. (5)

x	$0 < x < 3$	$3 < x$
$\frac{dA}{dx}$	< 0	> 0

(5)

$\therefore$ The surface area is minimum when $x = 3$. (5)

$y = \frac{45}{9} - \frac{6}{3}$

$= 3$ (5)

55

[*see page* 13

(c) $f(x) = \dfrac{a}{(x-1)^2} + \dfrac{b}{(x+1)}$

Since $f(0) = 2$,

$a + b = 2$ ——— (1) (5)

$f^{/}(x) = -\dfrac{2a}{(x-1)^3} - \dfrac{b}{(x+1)^2}$ (5)

Since $f^{/}(0) = 0$,

$2a - b = 0$ ——— (2) (5)

Form (1) and (2), $a = \dfrac{2}{3}$, $b = \dfrac{4}{3}$ (5)

$f^{/}(x) = -\dfrac{4}{3}\dfrac{1}{(x-1)^3} - \dfrac{4}{3(x+1)^2}$ (5)

$= -\dfrac{4}{3}\left\{\dfrac{(x+1)^2 + (x-1)^3}{(x-1)^3(x+1)^2}\right\}$

$= -\dfrac{4}{3}\left[\dfrac{x^3 - 2x^2 + 5x}{(x-1)^3(x+1)^2}\right]$

$= -\dfrac{4x}{3}\left[\dfrac{x^2 - 2x + 5}{(x-1)^3(x+1)^2}\right]$

$= -\dfrac{4x}{3}\left[\dfrac{(x-1)^2 + 4}{(x-1)^3(x+1)^2}\right]$

Since $(x-1)^2 + 4 > 0$ for all $x \in \mathbb{R}$, we have that $f^{/}(x) = 0$ if and only if $x = 0$. (5)

x	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
$f^{/}(x)$	< 0	< 0	> 0	< 0
	f decreases	f decreases	f increases	f decreases

(10)

The function f has a relative minimum at $x = 0$. (5)

Then $f(0) = 2$.

When $x \longrightarrow \pm\infty$ we have that $f(x) \longrightarrow 0$.

$x \longrightarrow -1^{-}$, $f(x) \longrightarrow -\infty$

$x \longrightarrow -1^{+}$, $f(x) \longrightarrow +\infty$

$x \longrightarrow 1^{-}$, $f(x) \longrightarrow +\infty$ (10)

$x \longrightarrow 1^{+}$, $f(x) \longrightarrow +\infty$

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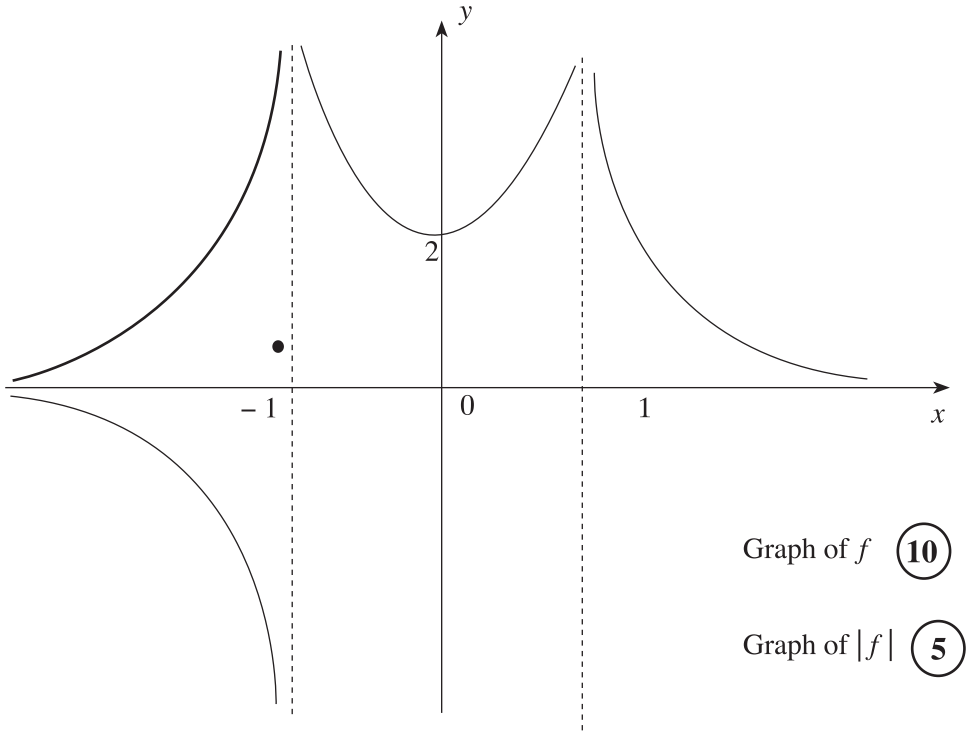


Graph of f (10)

Graph of $|f|$ (5)

70

15. (a) $I = \int_0^1 \frac{dx}{(2+x)^{1/2}(2-x)^{3/2}} = \int_0^1 \frac{dx}{(4-x^2)^{1/2}(2-x)}$

By substituting $x = 2\sin\theta$, (5)

$dx = 2\cos\theta \, d\theta$ (5)

$x = 0, \; \sin\theta = 0$

$\theta = 0$

$x = 1, \; \sin\theta = \frac{1}{2}$, $\theta = \frac{\pi}{6}$ (5)

$I = \int_0^{\pi/6} \frac{2\cos\theta}{(4 - 4\sin^2\theta)^{1/2}(2 - 2\sin\theta)} \, d\theta$ (5)

$= \int_0^{\pi/6} \frac{\cancel{2\cos\theta}}{\cancel{2\cos\theta}\; 2(1-\sin\theta)} \, d\theta$

$= \frac{1}{2}\int_0^{\pi/6} \frac{1 + \sin\theta}{\cos^2\theta} \, d\theta = \frac{1}{2}\int_0^{\pi/6} \sec^2 d\theta + \frac{1}{2}\int_0^{\pi/6} \sec\theta\tan\theta \, d\theta$ (5)

(5)

$= \frac{1}{2}\left[\tan\theta\right]_0^{\pi/6} + \frac{1}{2}\left[\sec\theta\right]_0^{\pi/6}$ (5)

[*see page* 15

$$= \frac{1}{2}\left[\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} - 1\right] = \frac{\sqrt{3}-1}{2}$$ (5) (5) (5)

50

(b) $G(x) = \dfrac{A}{x+2} + \dfrac{Bx+C}{x^2+8}$ (5)

$1 = A(x^2+8) + (Bx+C)(x+2)$ (5)

Coefficient of x^2 : $0 = A + B \Rightarrow A = -B$

Coefficient of x : $0 = 2B + C \Rightarrow C = -2B$ (5)

Constant: : $1 = 8A + 2C$ (5)

$1 = -8B - 4B \Rightarrow 12B = -1$

$\Rightarrow B = -\dfrac{1}{12}$ (5)

$A = \dfrac{1}{12}, \quad C = \dfrac{1}{6}$ (5) (5)

$$g(x) = \int \frac{1}{(x+2)(x^2+8)}\, dx$$

$$g(x) = \frac{1}{12}\int \frac{1}{(x+2)}\, dx - \frac{1}{12}\int \frac{x}{(x^2+8)}\, dx + \frac{1}{6}\int \frac{1}{(x^2+8)}\, dx$$ (5)

$$= \frac{1}{12}\ln|x+2| - \frac{1}{24}\ln(x^2+8) + \frac{1}{6}\tan^{-1}\left(\frac{x}{2\sqrt{2}}\right) + C$$ (5)

$$= \frac{1}{24}\ln\left[\frac{(x+2)^2}{x^2+8}\right] + \frac{1}{6}\,\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x}{2\sqrt{2}}\right) + C$$ (5) (5)

$$= \frac{1}{24}\ln\left[\frac{(x+2)^2}{x^2+8}\right] + \frac{1}{12\sqrt{2}}\tan^{-1}\left(\frac{x}{2\sqrt{2}}\right) + C$$ (5)

60

(c) $I_n = \int x^n \sin x \, dx$

$$= \int -x^n \frac{d}{dx}(\cos x)$$ (5)

$$= -x^n \cos x + \int (\cos x)\, n x^{n-1} dx$$ (10)

[*see page* 16

$$= -x^n \cos x + n \int x^{n-1} \frac{d}{dx} (\sin x)$$ (5)

$$= -x^n \cos x + n \left\{ x^{n-1} \sin x - \int \sin x (n-1)\, x^{n-2}\, dx \right\}$$ (10)

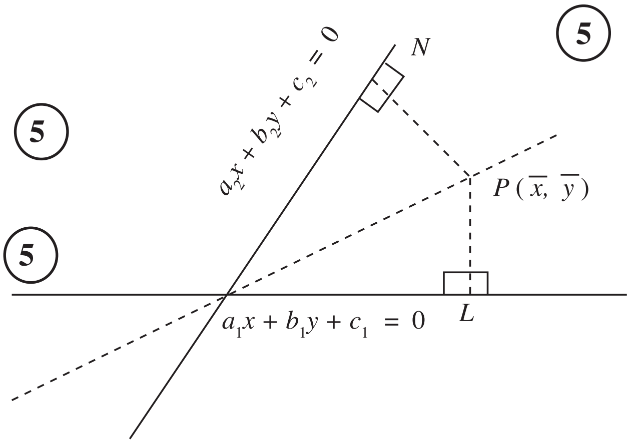
$$= -x^n \cos x + nx^{n-1} \sin x - n(n-1)\, I_{n-2}$$ (5)

$$I_n + n(n-1)\, I_{n-2} = x^{n-1}\left[n \sin x - x \cos x\right]$$ (5)

40

16. (a)

Let $P(\overline{x}, \overline{y})$ be an arbitrary point on any one of the angle bisectors.



(5)

$PL = PN$ (5)

$$\frac{|a_1\overline{x} + b_1\overline{y} + c_1|}{\sqrt{a_1^2 + b_2^2}} = \frac{|a_2\overline{x} + b_2\overline{y} + c_2|}{\sqrt{a_2^2 + b_2^2}}$$ (5)

$$\therefore \frac{a_1\overline{x} + b_1\overline{y} + c_1}{\sqrt{a_1^2 + b_2^2}} = \pm \frac{a_2\overline{x} + b_2\overline{y} + c_2}{\sqrt{a_2^2 + b_2^2}}$$ (5)

By replacing $\overline{x}$ by x, and $\overline{y}$ by y (5)

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_2^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$ (5)

The equations of the angle bisectors;

$$\frac{4x + y + 3}{\sqrt{4^2 + 1^2}} = \pm \frac{x + 4y - 3}{\sqrt{4^2 + 1^2}}$$ (5)

$+ :\ 3x - 3y + 6 = 0 \Rightarrow x - y + 2 = 0$

$- :\ 5x + 5y = 0 \Rightarrow x + y = 0$ (5)

By solving $x + y = 0$ and $x - y + 2 = 0$ we obtain

$x = -1, \quad y = 1$ (5)

Let $A = (-1, 1)$

$B = (0, 2)$ lies on the line given by $x - y + 2 = 0$. (5)

Let $P = (x, y)$ be a point on the line given by $x + y = 0$

Since PA is perpendicular to PB,

[see page 17

$\left(\frac{y-1}{x+1}\right)\times 1 = -1$ (5)

$\frac{y-1}{-1} = \frac{x+1}{1} = t$; t is a parameter. (5)

$\therefore\ x = -1+t,\quad y = 1-t$

Let T be the value of t corresponding to the point D which lies on $x + y = 0$ and is such that $AD = AB$.

Then $D = (-1+T, 1-T)$ (5)

$AD^2 = AB^2 \Rightarrow T^2 + T^2 = 1^2 + 1^2 = 2$ (5)

$T = \pm 1$ (5)

$\therefore\ D = (0, 0)$ or $(-2, +2)$

(5) (5)

When $D \equiv (0, 0)$ the equation of the side CD is

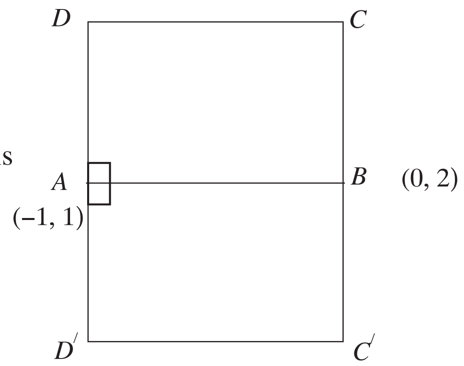
$x - y = 0$ (5)

When $D' \equiv (-2, +2)$ the equation of the side $C'D'$ is

$x - y + 4 = 0$ (5)

The equation of the side BC and the side BC' is

$x + y - 2 = 0$ (5)



100

(b) $S^1 = x^2 + y^2 - 2x + 4y - 3 = 0$

Let $S = x^2 + y^2 + 2gx + 2fy + c = 0$. Here g, f, c are constants. (5)

Since $S = 0$ is bisected by $S^1 = 0$, the centre of $S = 0$ lies on the line (5)
$S^1 - S = 0$ given by

$-2x(g+1) - 2y(f-2) - 3 - c = 0$ (5)

$\therefore\ 2(g)(g+1) + 2(f)(f-2) - c - 3 = 0$ ——— (1) (5)

Since the circle $S = 0$ passes through the point $(1, 1)$,

$1^2 + 1^2 + 2g + 2f + c = 0$

$\therefore\ c = -2g - 2f - 2$ ——— (2) (5)

[see page 18

From ① and ②,

$2g^2 + 2g + 2f^2 - 4f - (-2g - 2f - 2) - 3 = 0$ **(5)**

$2g^2 + 2f^2 + 4g - 2f - 1 = 0$ **(5)**

$2(-g)^2 + 2(-f)^2 - 4(-g) + 2(-f) - 1 = 0$

$\therefore$ the point $(-g, -f)$ lies on the circle $2x^2 + 2y^2 - 4x + 2y - 1 = 0$ **(5)**

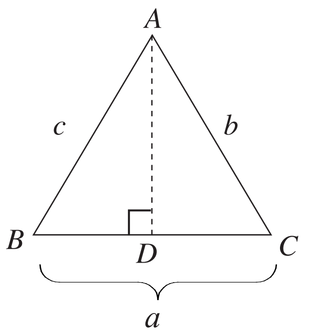
The centre of this circle is $(1, -\frac{1}{2})$ and its radius is **(5)**

$r = \sqrt{1^2 + \frac{1}{4} + \frac{1}{2}} = \sqrt{\frac{7}{4}}$

$= \frac{\sqrt{7}}{2}$ **(5)**

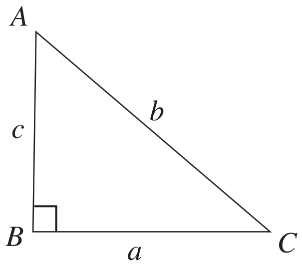
50

17. (a)



$a = BC = BD + DC$

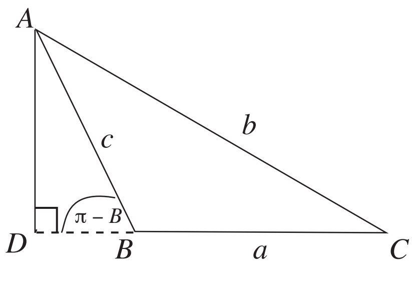
$a = c\cos B + b\cos C$ **(5)**



$a = b\cos C + 0$

$= b\cos C + c\cos 90°$ **(5)**

$= b\cos C + c\cos B$



$a = BC = CD - BD$

$= b\cos C - c\cos(\pi - B)$ **(5)**

$= b\cos C + c\cos B$

Similarly, $b = a\cos C + c\cos A$

$a\cos C = b - c\cos A$

$a^2\cos^2 C = b^2 - 2bc\cos A + c^2\cos^2 A$ **(10)**

$a^2 - a^2\sin^2 C = b^2 + c^2 - 2bc\cos A - c^2\sin^2 A$

[*see page* 19

$a^2 + \underbrace{c^2 \sin^2 A - a^2 \sin^2 C}_{=\ 0} = b^2 + c^2 - 2bc \cos A$; since $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$ (5)

$\therefore\ a^2 = b^2 + c^2 - 2bc \cos A$

$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

a, b, c are in an arithmetic progression. Therefore,

$a + c = 2b$ (5)

$b\cos C + c\cos B + a\cos B + b\cos A = 2b$ (5)

$\cos A + \cos C + 2\cos B = 2$

$2\cos\left(\dfrac{A+C}{2}\right)\cos\left(\dfrac{A-C}{2}\right) = 2(1-\cos B)$ (5)

$2\cos\left(\dfrac{\pi}{2} - \dfrac{B}{2}\right)\cos\left(\dfrac{A-C}{2}\right) = 4\sin^2\dfrac{B}{2}$

$\cos\left(\dfrac{A-C}{2}\right) = 2\sin\dfrac{B}{2}$ (5)

50

(b) $0 < x,\ y < \dfrac{\pi}{2}$

$\therefore\ 0 < \dfrac{\pi}{2} - y < \dfrac{\pi}{2}$ (5)

$\sin x > \cos y = \sin\left(\dfrac{\pi}{2} - y\right)$ (5)

$\sin x > \sin\left(\dfrac{\pi}{2} - y\right)$

As the angle increases in the domain $\left(0, \dfrac{\pi}{2}\right)$, the sine value also increases. (10)
Therefore,

$\therefore\ x > \dfrac{\pi}{2} - y$ (5)

$x + y > \dfrac{\pi}{2}$

25

(c) $f(x) = 3\cos^2 x + 8\sin x\cos x - 3\sin^2 x$

$= 3\cos 2x + 4\sin 2x$ (5) (5)

$= 5\left(\dfrac{3}{5}\cos 2x + \dfrac{4}{5}\sin 2x\right)$ (5)

$= 5(\sin\alpha\cos 2x + \cos\alpha\sin 2x)$

$= 5\sin(2x + \alpha)$

[*see page* 20

$= \quad A\sin(2x+\alpha)$ (5)

Here $A = 5$, and α is an acute angle such that $\tan\alpha = \frac{3}{4}$. (5)

$f(x) = \frac{5}{2}$

$5\sin(2x+\alpha) = \frac{5}{2}$

$\sin(2x+\alpha) = \frac{1}{2} = \sin\frac{\pi}{6}$ (5)

$2x+\alpha = n\pi + (-1)^n \frac{\pi}{6}$ (5)

$x = \frac{n\pi}{2} - \frac{\alpha}{2} + (-1)^n\frac{\pi}{12}$, here $n \in \mathbb{Z}$

(5)

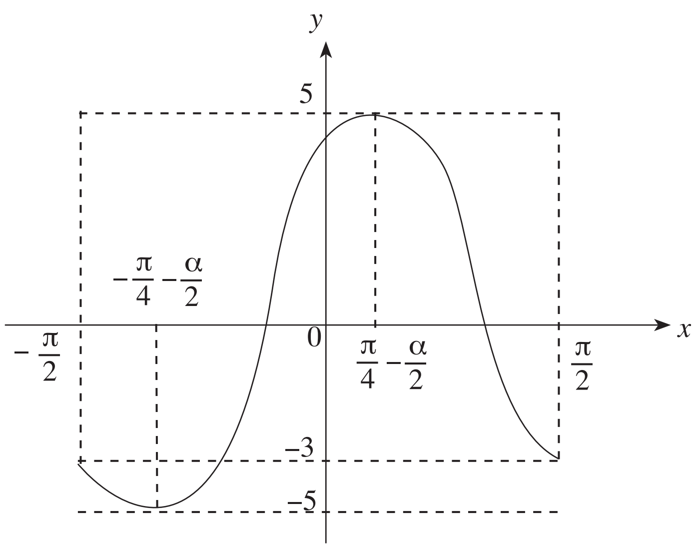
$f(x) = 5\sin(2x+\alpha)$

Maximum $f(x) = 5$; $x = \frac{\pi}{4} - \frac{\alpha}{2}$

(5) (5)

Minimum $f(x) = -5$; $x = -\frac{\pi}{4} - \frac{\alpha}{2}$ (Since $\alpha < \frac{\pi}{4}$)

(5) (5)



(15)

75

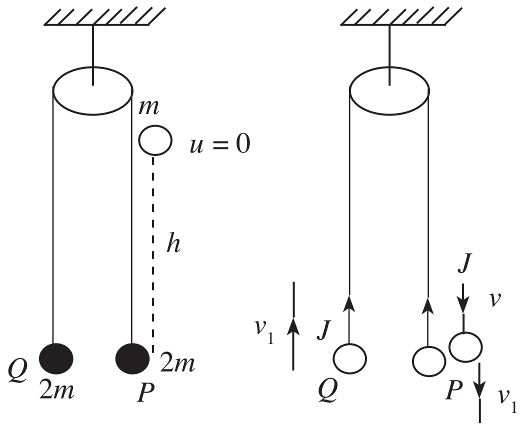
G.C.E. (A.L.) Support Seminar - 2016
Combined Mathematics - Paper II
Answer Guide

Part A

1. Applying $v^2 = u^2 + 2as$ to $m \downarrow$

$v^2 = 2gh$

$\therefore v = \sqrt{2gh}$ (5)



Applying $I = \Delta(mv)$;

To P and $m \downarrow$

$-J = (2m + m)v_1 - mv - 2m \times 0$ (5)

$\therefore -J = 3mv_1 - mv$ ——— (1)

To $Q \uparrow$

$J = 2mv_1 - 0$ ——— (2) (5)

From (1) and (2),

$v_1 = \frac{v}{5} = \frac{\sqrt{2gh}}{5}$ (5)

$J = \frac{2m}{5}\sqrt{2gh}$ (5)

25

2. The volume of water that is ejected in a second $= 8\,(0.005)\ \text{m}^3$

$= 0.040\ \text{m}^3$ (5)

The mass of water that is ejected in a second $= 10^3 \times 0.040$ kg

$= 40$ kg (5)

Work done by the pump in a second $= mgh + \frac{1}{2}mv^2$

$= (40 \times 10 \times 4) + \frac{1}{2} \times 40 \times 8^2$
(5) (5)

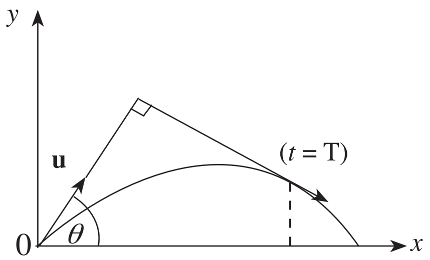
$= 2880\ \text{js}^{-1}$

$\therefore$ the power of the pump $= 2880$ W (5)

25

[See page 2

3.

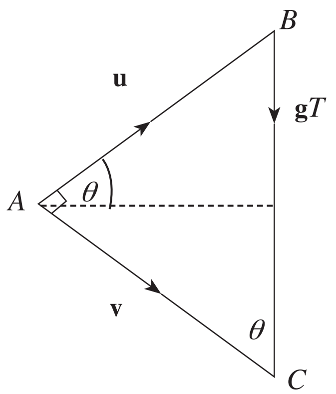


(5)

When $t = T$

$\mathbf{v} = \mathbf{u} + \mathbf{g}T$ (5)

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$



(10)

$gT \sin\theta = u$ (5)

$\therefore \quad T = \dfrac{u}{g \sin\theta}$

$= \dfrac{u}{g} \operatorname{cosec}\theta$

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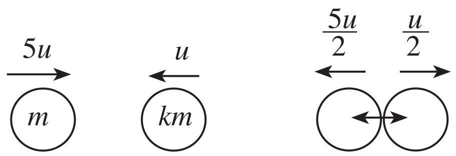
4.



By apply the law of conservation of momentum to the system

$\longrightarrow \quad 5mu - kmu = \dfrac{kmu}{2} - \dfrac{5mu}{2}$ (5)

$10 - 2k = k - 5$

$\therefore k = 5$ (5)

From Newton's law of restitution

$\dfrac{u}{2} + \dfrac{5u}{2} = e(u + 5u)$ (5)

$3u = 6ue$

$\dfrac{1}{2} = e$ (5)

$\underline{I} = \Delta(m\underline{v})$

$\longrightarrow \quad -I = -m.\dfrac{5u}{2} - m.5u$

$I = \dfrac{15mu}{2}$ (5)

25

[See page 3

5. Since $\underline{a} \perp \underline{b}$, we have $\underline{a} \cdot \underline{b} = 0$ (5)

$\therefore \quad (2\mathbf{i} + 3\mathbf{j}) \cdot (\lambda\mathbf{i} + \mu\mathbf{j}) = 0$

$2\lambda + 3\mu = 0$ ———— (1) (5)

Since $|\underline{b}| = 1$, we have $\lambda^2 + \mu^2 = 1$ ——— (2) (5)

From (1) and (2), $\mu = \pm\dfrac{2}{\sqrt{13}}$

Since $\mu > 0, \mu = \dfrac{2}{\sqrt{13}}$ (5)

From (1), $\lambda = -\dfrac{3}{\sqrt{13}}$ (5)

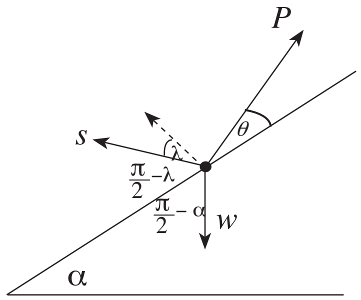
25

6. The object is in limiting equilibrium.

By Lami's Theorem,

$$\frac{P}{\sin[\pi - (\alpha+\lambda)]} = \frac{w}{\sin[\frac{\pi}{2} - (\theta - \lambda)]}$$ (5)

$$P = \frac{w \sin(\lambda + \alpha)}{\cos(\theta - \lambda)}$$ (5)



(5)

For P to be minimum, $(\theta - \lambda)$ should be maximum.

That is, $\theta = \lambda$ (5)

$\therefore P\text{(minimum)} = w \sin(\lambda + \alpha)$ (5)

25

7. Let $P(A) = \dfrac{1}{3}$ and $P(B) = \dfrac{1}{4}$.

$(A) \longrightarrow 1^{st}$ (First) $\quad (B) \longrightarrow 2^{nd}$ (Second)

(i) $X = (A \cap B') \cup (A' \cap B)$ (5)

But $(A \cap B') \cap (A' \cap B) = \phi$

$\therefore P(X) = P(A \cap B') + P(A' \cap B)$ ($\because$ By Axiom III)

$= P(A)\, P(B') + P(A')\, P(B)$ (5) (Since the events are independent)

$= \dfrac{1}{3} \times \left(1 - \dfrac{1}{4}\right) + \left(1 - \dfrac{1}{3}\right) \cdot \dfrac{1}{4}$

$= \dfrac{1}{4} + \dfrac{2}{3} \times \dfrac{1}{4} = \left(\dfrac{1}{4} \times \dfrac{5}{3}\right) = \dfrac{5}{12}$ (5)

(ii) $P(A \mid X) = \dfrac{P(A \cap X)}{P(X)} = \dfrac{P(A)\, P(B')}{P(X)}$ (5)

$= \dfrac{\frac{1}{3} \times \frac{3}{4}}{\frac{5}{12}} = \dfrac{3}{5}$ (5)

25

[See page 4

8. $P(A \cap B') = 0.2, \quad P(A' \cap B) = 0.1$ (5)

$P(A' \cap B') = P(A \cup B)' = 0.6$

$1 - P(A \cup B) = 0.6$

$P(A \cup B) = 0.4$ (5)

$P(A \cup B) - P(A \cap B) = 0.2 + 0.1$

$\therefore P(A \cap B) = 0.4 - 0.3 = 0.1$ (5)

$P(A' \cap B) = P(B) - P(A \cap B)$

$0.1 + 0.1 = P(B)$ (5)

$\therefore P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.1}{0.2}$

$= \dfrac{1}{2}$ (5)

25

9. $\overline{x} = 5$ and $s_x = 2$

(i) $y_i \in \{12, 13, 14, 15, 16, 17, 18\}$

Let $y_i = x_i + 10$.

Here $x_i \in \{2, 3, 4, 5, 6, 7, 8\}$

$\therefore \overline{y} = \overline{x} + 10 = 5 + 10 = 15$ (5)

and $s_y = s_x = 2$

(ii) $y_i \in \{20, 30, 40, 50, 60, 70, 80\}$

Let $y_i = 10x_i$

Here $x_i \in \{2, 3, 4, 5, 6, 7, 8\}$

$\therefore \overline{y} = 10\,\overline{x}$ (5)

$= 10 \times 5 = 50$ (5)

and $s_y = 10\,s_x = 10 \times 2 = 20$

(iii) Let $y_i = ax_i + b$.

Then $\overline{y} = a\overline{x} + b = 5a + b$ (5)

$s_y^2 = a^2 s_x^2$

$s_y = |a|\, s_x$

$= 2|a|$ (5)

25

10.

u_i	-3	-2	-1	0	1	2
f_i	5	10	25	30	20	10
$f_i u_i$	-15	-20	-25	0	20	20

(5)

$\overline{u} = \dfrac{\Sigma f_i u_i}{\Sigma f_i} = -\dfrac{20}{100} = -\dfrac{1}{5}$

$u_i = \dfrac{x_i - 35}{a}$ (5)

$\therefore \overline{x} = a\overline{u} + 35$

$33 = -\dfrac{a}{5} + 35$ (5)

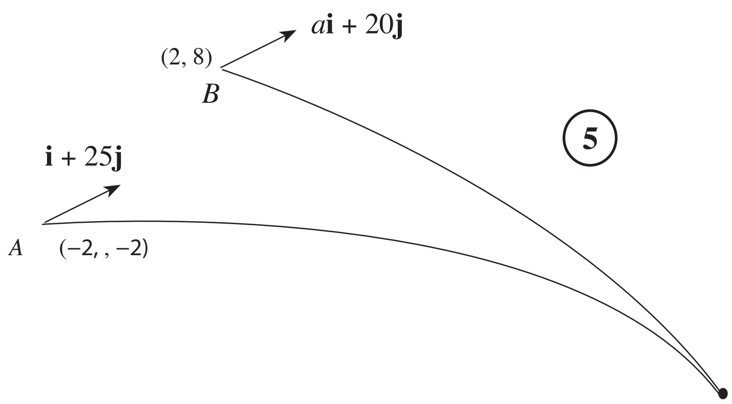
$a = 10$ (5)

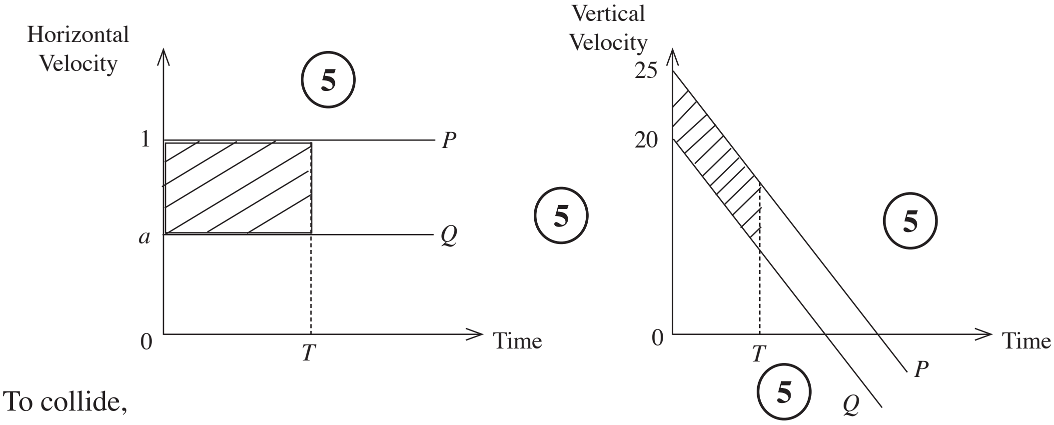
Intervals	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60
f_i	5	10	25	30	20	10

(5)

25

11. (a)





To collide,

Vertical displacement of P = Vertical displacement of Q + 10

Vertical displacement of P – Vertical displacement of Q = 10 (10)

[See page 6

$5T = 10$

$T = 2$ (5)

Horizontal displacement of P = Horizontal displacement of $Q + 4$ (10)

Horizontal displacement of P − Horizontal displacement $Q = 4$

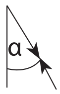
$(1 - a)\,2 = 4$ (5)

$1 - a = 2$

$a = -1$ (5)

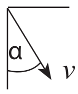
60

(b) $v\,(S, E) = \longrightarrow u$

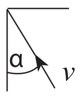
$v\,(P, S) =$  v (5)

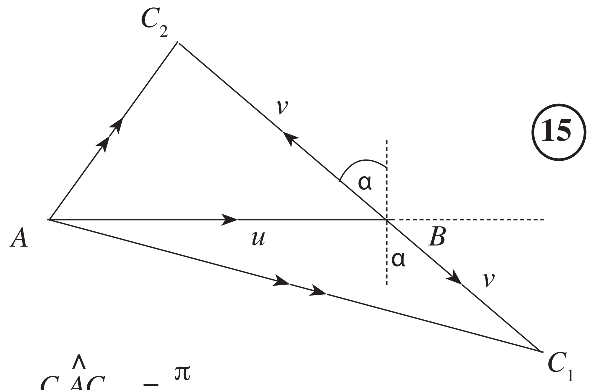
$v\,(P, E) = v\,(P, S) + v\,(S, E)$

Out

$\underline{v}\,(P, E) =$  $+ \longrightarrow u \Rightarrow \overrightarrow{BC_1} + \overrightarrow{AB} = \overrightarrow{AC_1}$ (5)

In

$\underline{v}\,(P, E) =$  $+ \longrightarrow u \Rightarrow \overrightarrow{BC_2} + \overrightarrow{AB} = \overrightarrow{AC_2}$ (5)



(15)

$C_1\hat{A}C_2 = \frac{\pi}{2}$

Hence, the circle with diameter C_1C_2 passes through the point A.

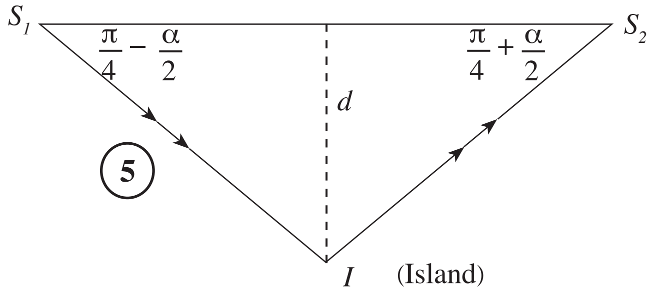
Furthermore, since the midpoint of C_1C_2 is B, (10)

$BC_1 = BC_2 = BA = u$ (5)

$v = u$

$B\hat{A}C_1 = \frac{\pi}{4} - \frac{\alpha}{2}$ (5) and $B\hat{A}C_2 = \frac{\pi}{4} + \frac{\alpha}{2}$ (5)

If the total time for the journey is t, (5)



$$t = \frac{S_1I}{AC_1} + \frac{S_2I}{AC_2}$$ (10)

$$= \frac{S_1I\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}{AC_1\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)} + \frac{S_2I\sin\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)}{AC_2\sin\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)}$$ (10)

$$= \frac{d}{v\cos\alpha} + \frac{d}{v\cos\alpha} = \frac{2d}{u\cos\alpha} \quad (\because v = u)$$

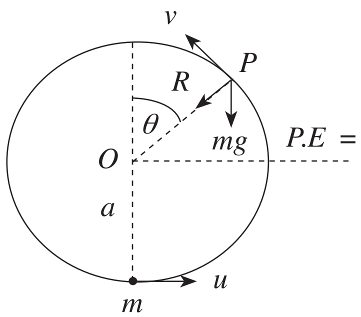
(5) (5)

90

12. (a) By the law of conservation of energy;

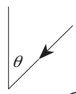
$$\frac{1}{2}mu^2 - mga = mga\cos\theta + \frac{1}{2}mv^2$$ (15)

$$v^2 - u^2 + 2ga(1+\cos\theta) = 0 \quad \text{—— (1)}$$ (5)



(5)

Applying $F = ma$ to m



$$R + mg\cos\theta = \frac{mv^2}{a} \quad \text{—— (2)}$$ (10)

By substituting from (1) into (2)

$$R + mg\cos\theta = \frac{m}{a}\left[u^2 - 2ga(1+\cos\theta)\right]$$ (5)

$$R = \frac{mu^2}{a} - mg(2 + 3\cos\theta)$$

If the particle leaves the surface when OA makes an angle of α with the upward vertical, then at that point $R = 0$. (5)

$$\therefore\ u^2 - 2ga - 3ga\cos\alpha = 0$$ (5)

$$\cos\alpha = \frac{u^2 - 2ga}{3ga} > 0 \quad (\because u^2 > 2ga)$$

$\therefore\ \alpha$ is an acute angle.

Furthermore, since $0 < \alpha < \frac{\pi}{2}$, we have that $0 < \cos\alpha < 1$.

$$\frac{u^2 - 2ga}{3ga} < 1$$ (5)

$$u^2 < 5ga$$

[See page 8

When the particle of mass m leaves the surface, $\cos\alpha = \dfrac{1}{\sqrt{3}}$.

$$\frac{1}{\sqrt{3}} = \frac{u^2 - 2ga}{3ga}$$

$$u^2 - 2ga = \sqrt{3}\,ga$$

$$u^2 = (2+\sqrt{3})\,ga \quad (5)$$

Then the velocity $v^2 = u^2 - 2ga\left(1+\dfrac{1}{\sqrt{3}}\right) = 2ga + \sqrt{3}\,ga - 2ga - \dfrac{2ga}{\sqrt{3}} = \dfrac{ga}{\sqrt{3}}$ (5)

After the particle of mass m leaves the surface of the sphere, its motion is that of a projectile.

In the ensuing motion, if the time taken to travel a horizontal distance $a\sin\alpha$ is t_0.

$$a\sin\alpha = (v\cos\alpha)\,t_0 \quad (5)$$

Then the distance traveled upward $y = (v\sin\alpha)t_0 - \dfrac{1}{2}g t_0^2$

$$y = \frac{v\sin\alpha \times a\sin\alpha}{v\cos\alpha} - \frac{1}{2}\,\frac{ga^2\sin^2\alpha}{v^2\cos^2\alpha} \quad (5)$$

$$= \frac{\frac{2a}{3}}{\frac{1}{\sqrt{3}}} - \frac{ga^2}{\frac{2ga}{\sqrt{3}}}\;\frac{\frac{2}{3}}{\frac{1}{3}}$$

$$= \frac{2a}{\sqrt{3}} - \sqrt{3}\,a$$

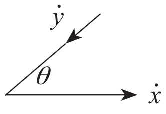
$$= -\frac{a}{\sqrt{3}}$$

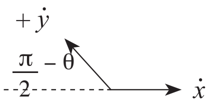
$$= -a\cos\alpha \quad (5)$$

Since the particle of mass m has travelled a distance of $a\cos\alpha$ downward when it passes the vertical line through O, it passes through the centre O of the sphere.

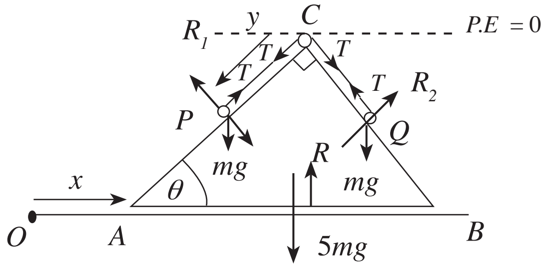
80

(b)

$\mathbf{v}(P, O) =$ 

$\mathbf{v}(Q, O) =$ 

(5)



Applying the law of conservation of momentum;

$\longrightarrow \quad 5m\dot{x} + m(\dot{x} - \dot{y}\cos\theta) + m(\dot{x} - \dot{y}\sin\theta) = 0$ (10)

$$7m\dot{x} = m\dot{y}(\cos\theta + \sin\theta)$$

$$7\dot{x} = \dot{y}\left(\frac{3}{5} + \frac{4}{5}\right)$$ (5)

$$5\dot{x} = \dot{y} \quad \text{———} \quad (1)$$

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Applying the law of conservation of energy;

$$\frac{1}{2}\,5m\,\dot{x}^2 + \frac{1}{2}\,m\,\{(\dot{x} - \dot{y}\cos\theta)^2 + (\dot{y}\sin\theta)^2\}$$ (20)

$$+ \frac{1}{2}\,m\,\{(\dot{x} - \dot{y}\sin\theta)^2 + (\dot{y}\cos\theta)^2\} - mgy\sin\theta - mg(l-y)\cos\theta = \text{constant}$$

$$5\dot{x}^2 + \left\{\dot{x}^2 + \dot{y}^2 - 2\dot{x}\dot{y}\cos\theta\right\} + \left\{\dot{x}^2 + \dot{y}^2 - 2\dot{x}\dot{y}\sin\theta\right\}$$

$$-2gy\sin\theta + 2gy\cos\theta = \text{constant}$$ (5)

$$7\dot{x}^2 + 2\dot{y}^2 - 2\dot{x}\dot{y}\left(\frac{4}{5} + \frac{3}{5}\right) - 2gy\,\frac{4}{5} + 2gy\,\frac{3}{5} = \text{constant}$$

$$35\dot{x}^2 + 10\dot{y}^2 - 14\dot{x}\dot{y} - 2gy = \text{constant} \quad \text{———} \quad (2)$$

25

From (1) and (2)

$$35\dot{x}^2 + 250\dot{x}^2 - 70\dot{x}^2 - 2gy = \text{constant}$$

$$215\dot{x}^2 - 2gy = \text{constant}$$

Differentiating with respect to t,

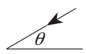
$$430\dot{x}\cdot\ddot{x} - 2g\dot{y} = 0$$ (5)

$$430\dot{x}\cdot\ddot{x} - 2g\,.\,5\dot{x} = 0 \quad (\therefore\ \dot{x} \neq 0)$$ (5)

$$\therefore\ \ddot{x} = \frac{g}{43}$$

10

Applying $F = ma$ to P



$$: mg\sin\theta - T = m(\ddot{y} - \ddot{x}\cos\theta)$$ (5)

$$T = mg\sin\theta - m(5\ddot{x} - \ddot{x}\cos\theta)$$ (5)

$$= mg\,\frac{4}{5} - m\ddot{x}\left(5 - \frac{3}{5}\right)$$

$$= \frac{4mg}{5} - m.\frac{1}{43}g\,.\,\frac{22}{5}$$

$$= \frac{2mg}{5}\left\{2 - \frac{11}{43}\right\}$$

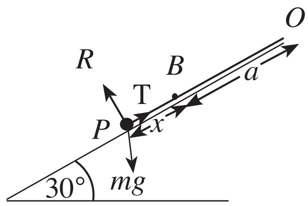
$$= \frac{2mg}{5} \times \frac{75}{43}$$

$$= \frac{30mg}{43}$$ (5)

15

[See page 10

13. If the tension in the string when it is extended a length x from its natural length is T, then



$T = \frac{\lambda x}{a} = \frac{2mgx}{a}$ (5)

Applying $F = ma$ to the motion of the particle;

$mg \sin 30° - T = m\ddot{x}$ (10)

$mg \times \frac{1}{2} - \frac{2mgx}{a} = m\ddot{x}$

$\ddot{x} = -\frac{2g}{a}\left(x - \frac{a}{4}\right)$ ——— (1) (5)

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$x = \frac{a}{4} + A \cot \omega t + B \sin \omega t$ ——— (2)

$\dot{x} = -A\omega \sin \omega t + B\omega \cos \omega t$ ——— (3) (5)

$\ddot{x} = -A\omega^2 \cos \omega t - B\omega^2 \sin \omega t$ ——— (4) (5)

$= -\omega^2 (A \cos \omega t + B\omega \sin \omega t)$

$\ddot{x} = -\omega^2 \left(x - \frac{a}{4}\right)$ ——— (5) by (2) (5)

By considering (1) and (5) $\omega^2 = \frac{2g}{a}$. (5)

$\omega = \sqrt{\frac{2g}{a}}$

$t = 0$ when $\dot{x} = 0$. (5)

From (3), $0 = B\omega$

Since $\omega \neq 0$ we have that $B = 0$. (5)

$x = a$ when $t = 0$ (5)

40

From (2), $a - \frac{a}{4} = A \Rightarrow A = \frac{3a}{4}$ (5)

$\therefore x = \frac{3a}{4} \cos \omega t + \frac{a}{4}$

$x - \frac{a}{4} = \frac{3a}{4} \cos \omega t$

The centre of the oscillation is given by $x - \frac{a}{4} = 0$. (5)

That is, $x = \frac{a}{4}$ is the centre. (5)

10

[See page 11

When $\dot{x} = 0$ at the amplitude, let $t = t_1$.

$0 = -A\omega \sin \omega t_1$ (5)

$\sin \omega t_1 = 0$

$\omega t_1 = n\pi \; ; \; n \in \mathbb{Z}_0^+$ (5)

$x - \frac{a}{4} = \frac{3a}{4} \cos \omega t_1$

$x - \frac{a}{4} = \pm \frac{3a}{4}$ (5)

20

$\therefore$ the amplitude of the simple harmonic motion of the particle $= \frac{3a}{4}$ (5)

Let the velocity of the particle when it first arrives at the natural length of the string be V.

Then $x = 0$. (5)

$\frac{3a}{4} \cos \omega t = -\frac{a}{4}$

$\cos \omega t = -\frac{1}{3}$ (5)

$V = -A\omega \sin \omega t$

$= -\frac{3a}{4} \sqrt{\frac{2g}{a}} \cdot \sqrt{1 - \cos^2 \omega t}$

$= -\frac{3a}{4} \sqrt{\frac{2g}{a}} \sqrt{\frac{8}{9}}$ (5) $= -\frac{3a}{4} \sqrt{\frac{2g}{a}} \cdot \frac{2\sqrt{2}}{3}$

$= -\sqrt{ag}$

The velocity of the particle when $x = 0$ is $\uparrow \sqrt{ag}$. (5)

20

Let t_0 be the time when the particle first arrives at the natural length of the string.

Then $x = 0$. (5)

$-\frac{a}{4} = \frac{3a}{4} \cos \omega t_0$

$\cos \omega t_0 = -\frac{1}{3}$ (5)

$\omega t_0 = \pi - \cos^{-1} (\frac{1}{3})$

$t_0 = \frac{1}{\omega} \left[\pi - \cos^{-1} (\frac{1}{3})\right] = \sqrt{\frac{a}{2g}} \left[\pi - \cos^{-1} (\frac{1}{3})\right]$ (5)

The particle travels up to the point O under gravity.

If the time taken for the particle to travel from B to O is t_2 ,

[See page 12

$S = ut + \frac{1}{2}at^2$ ↑

$S = a,\ u = \sqrt{ag},\ a = -g\sin 30°$

$a = \sqrt{ag}\,t_2 - \frac{1}{2}\,\frac{g}{2}\,t_2^{\,2}$ (5)

$\frac{g}{4}t_2^{\,2} - \sqrt{ag}\,t_2 + a = 0$

$$t_2 = \frac{\sqrt{ag} \pm \sqrt{ag - 4\frac{g}{4}a}}{\frac{g}{2}}$$

$t_2 = 2\sqrt{\frac{a}{g}}$ (5)

$\therefore$ the time taken to travel to $O = t_0 + t_2$,

$= \sqrt{\frac{a}{2g}}\left(\pi - \cos^{-1}\left(\frac{1}{3}\right)\right) + 2\sqrt{\frac{a}{g}}$ (5)

$= \sqrt{\frac{a}{2g}}\left[\pi - \cos^{-1}\left(\frac{1}{3}\right) + 2\sqrt{2}\right]$ **30**

The string is at its maximum length at A ; that is when $x = a$,

$T_A = \frac{\lambda a}{a}$ (5) **10**

$= \lambda$

$T_A = 2mg$ (5)

14. (a) (i) $|\underline{a}| = |\underline{b}| = |\underline{c}| = 1$ (5)

If $(\underline{a} + 2\underline{b}) \perp (5\underline{a} - 4\underline{b})$,

$(\underline{a} + 2\underline{b}) \cdot (5\underline{a} - 4\underline{b}) = 0$ (5)

$5\underline{a}\cdot\underline{a} + 10\underline{b}\cdot\underline{a} - 4\underline{a}\cdot\underline{b} - 8\underline{b}\cdot\underline{b} = 0$

$5|\underline{a}|^2 + 10\underline{a}\cdot\underline{b} - 4\underline{a}\cdot\underline{b} - 8|\underline{b}|^2 = 0$

$5 + 6\underline{a}\cdot\underline{b} - 8 = 0$

$6\underline{a}\cdot\underline{b} = 3$

$\underline{a}\cdot\underline{b} = \frac{1}{2}$ (5)

$|\underline{a}|\,|\underline{b}|\cos\theta = \frac{1}{2}$ (5)

$1 \times 1\ \cos\theta = \frac{1}{2} \Rightarrow \theta = 60°$ (5) **25**

[See page 13

(ii) $|\underline{a}-\underline{b}|^2 + |\underline{b}-\underline{c}|^2 + |\underline{c}-\underline{a}|^2$

$= (\underline{a}-\underline{b}).(\underline{a}-\underline{b}) + (\underline{b}-\underline{c}).(\underline{b}-\underline{c}) + (\underline{c}-\underline{a}).(\underline{c}-\underline{a})$ (5)

$= |\underline{a}|^2 + |\underline{b}|^2 - 2\underline{a}.\underline{b} + |\underline{b}|^2 + |\underline{c}|^2 - 2b.\underline{c} + |\underline{c}|^2 + |\underline{a}|^2 - 2\underline{c}.\underline{a}$ (5)

$= 6 - 2(\underline{a}.\underline{b} + \underline{b}.\underline{c} + \underline{c}.\underline{a})$ (5)

$\therefore 2(\underline{a}.\underline{b} + \underline{b}.\underline{c} + \underline{c}.\underline{a}) = 6 - \left(|\underline{a}-\underline{b}|^2 + |\underline{b}-\underline{c}|^2 + |\underline{c}-\underline{a}|^2\right)$ —— (1) (5)

$|\underline{a}+\underline{b}+\underline{c}|^2 \geq 0$ (5)

$\therefore (\underline{a}+\underline{b}+\underline{c}).(\underline{a}+\underline{b}+\underline{c}) \geq 0$ (5)

$|\underline{a}|^2 + \underline{a}.\underline{b} + \underline{a}.\underline{c} + \underline{b}.\underline{a} + |\underline{b}|^2 + \underline{b}.\underline{c} + \underline{c}.\underline{a} + \underline{c}.\underline{b} + |\underline{c}|^2 \geq 0$ (5)

$3 + 2(\underline{a}.\underline{b} + \underline{b}.\underline{c} + \underline{c}.\underline{a}) \geq 0$ —— (2) (5)

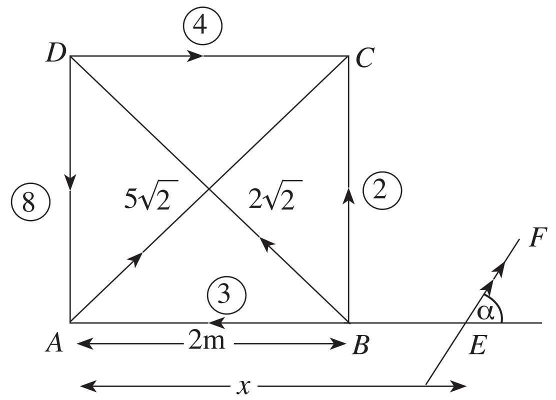
From (1) and (2)

$3 + 6 - (|\underline{a}-\underline{b}|^2 + |\underline{b}-\underline{c}|^2 + |\underline{c}-\underline{a}|^2) \geq 0$ (5)

$\therefore \quad |\underline{a}-\underline{b}|^2 + |\underline{b}-\underline{c}|^2 + |\underline{c}-\underline{a}|^2 \leq 9$ (5)

50

(b)



(5)

(i) $\overrightarrow{X} = 4 - 3 + 5\sqrt{2}\cos 45° - 2\sqrt{2}\cos 45°$ (5)

$= 4\text{N}$

$\uparrow Y = 2 - 8 + 5\sqrt{2}\cos 45° + 2\sqrt{2}\cos 45°$ (5)

$= 1\text{N}$

If the resultant is R,

$R = \sqrt{X^2 + Y^2} = \sqrt{4^2 + 1^2}$

$= \sqrt{17}\,\text{N}$ (5)

If the resultant makes an angle α with the horizontal,

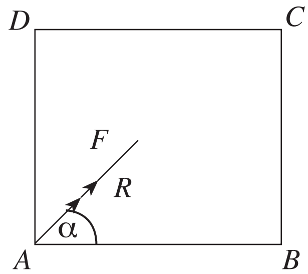
$\tan\alpha = \frac{1}{4}$

$\alpha = \tan^{-1}\left(\frac{1}{4}\right)$ (5)

25

[See page 14

If the point at which the line of action of the resultant intersects AB is E, let $AE = x$.



$\overset{\curvearrowleft}{A}$ $\quad 1 \times x = 2 \times 2 - 4 \times 2 + 2\sqrt{2} \,.\, 2\cos 45^\circ$ (5)

$x = 0$ (5)

$A \equiv E$ (Coincides)

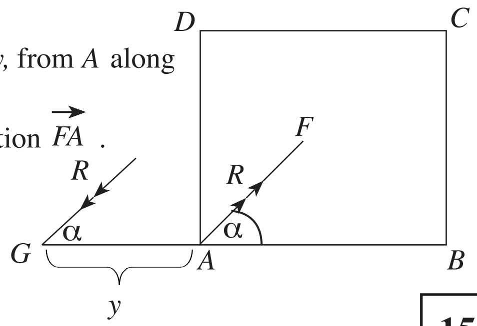
The line of action of the resultant passes through A. (5)

$\therefore$ for the system of forces to be in equilibrium, a force of $\sqrt{17}$ N should be introduced at A in the direction $\overrightarrow{FA}$. (5)

20

(ii) To reduce the system to a couple of magnitude 39Nm acting in the sense ABC, (5)

a force of $\sqrt{17}$ N should be introduced at a distance, say y, from A along BA produced, such that it is parallel to AF and in the direction $\overrightarrow{FA}$.



$\overset{\curvearrowleft}{ABC}$ $\quad \sqrt{17} \times AG \sin\alpha = 39$ (5)

$\sqrt{17} \times AG \times \dfrac{1}{\sqrt{17}} = 39$

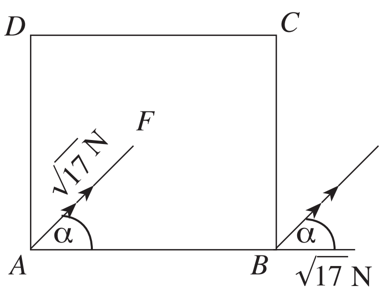
$AG = 39\text{m}$ (5)

15

(iii) The couple that should be introduced to reduce the system to a single force acting at B; (5)

$= \curvearrowleft \sqrt{17} \times BA \sin\alpha$ (5)

$= \curvearrowleft \sqrt{17} \times 2 \times \dfrac{1}{\sqrt{17}} = 2\text{Nm}$ (5)



Aliter

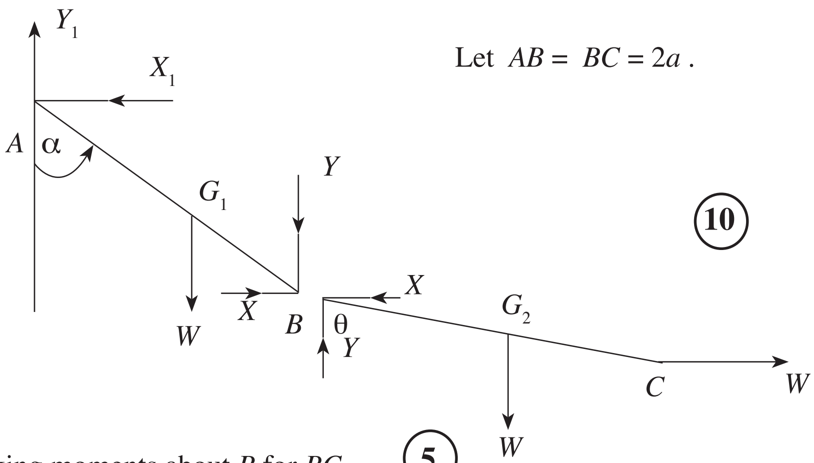
If the couple that should be introduced is M,

$M - \sqrt{17} \times 2\sin\alpha = 0$

$M = \sqrt{17} \times 2 \times \dfrac{1}{\sqrt{17}} = 2 \text{ Nm}$

15

15. (a)



Let $AB = BC = 2a$.

(10)

10

(i) Taking moments about B for BC (5)

$W a \sin\theta = 2Wa\cos\theta$

$\tan\theta = 2$ (5)

10

Considering the equilibrium of BC,

$\leftarrow \; X = W$ (5)

$\uparrow \; Y = W$ (5)

$\therefore \; R_B = \sqrt{W^2 + W^2}$ (5)

$= \sqrt{2}\, W$

20

R_B makes an angle of $\tan^{-1} 1 = \frac{\pi}{4}$ with the horizontal (5)

AB ⊃ ↶ A

$X.\, 2\cancel{a}\cos\alpha = W\cancel{a}\sin\alpha + y.\, 2\cancel{a}\sin\alpha$ (10)

$W\; 2\cos\alpha = W\sin\alpha + W.\, 2\sin\alpha$

$\frac{2}{3} = \tan\alpha$

$\alpha = \tan^{-1}\left(\frac{2}{3}\right)$ (5)

15

By considering the equilibrium of AB ,

$\rightarrow \; X_1 = X = W$ (5)

$\uparrow \; Y_1 = 2W$ (5)

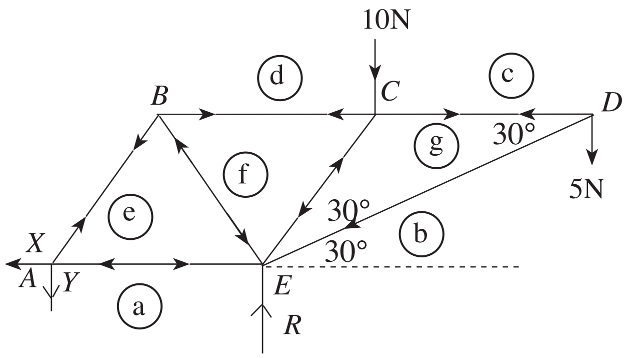
$\therefore \; R_A = \sqrt{X_1^2 + Y_1^2}$

$= \sqrt{5}\, W$ (5)

R_A makes an angle of $\tan^{-1}(2)$ with the horizontal (5)

20

(b)



Let the length of a rod (other than DE) be $2a$.

(i) Considering the system, ↶ A

$R\, 2\cancel{a} - 10 \times 3\cancel{a} - 5 \times 5\cancel{a} = 0$ (5)

$R = \frac{55}{2}$ N (5)

$\therefore$ the vertical force acting at $E = \frac{55}{2}$ N

[See page 16

(ii) $\uparrow \; -Y + R - 10 - 5 = 0$

$-Y = 15 - \frac{55}{2} = -\frac{25}{2}$ N

$\therefore \; Y = \frac{25}{2}$ N

$\leftarrow \; X = 0$

The vertical component of the reaction at the hinge A = $\downarrow \frac{25}{2}$ N (5)

Horizontal component = 0 (5)

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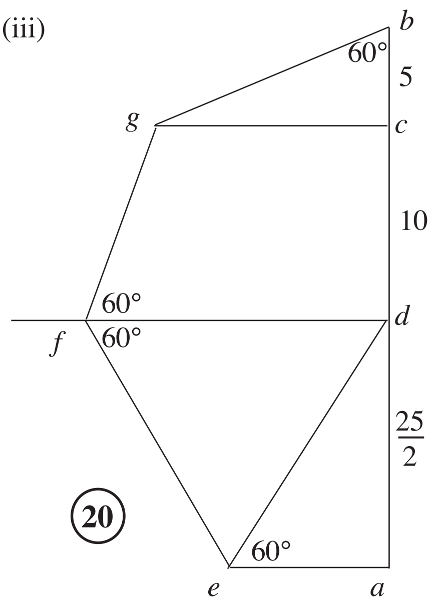
(iii)

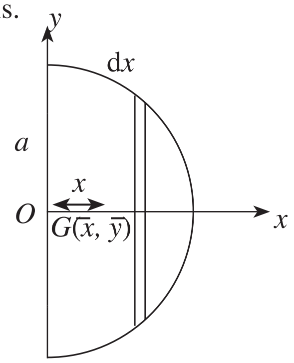


Rod	Magnitude	Stress
AE	$\frac{25\sqrt{3}}{6}$ N	Thrust
AB	$\frac{25\sqrt{3}}{3}$ N	Tension
BE	$\frac{25\sqrt{3}}{3}$ N	Thrust
BC	$\frac{25\sqrt{3}}{3}$ N	Tension
CE	$\frac{20\sqrt{3}}{3}$ N	Thrust
CD	$5\sqrt{3}$	Tension
ED	10 N	Thrust

(35)

55

16. By symmetry, the centre of mass lies on the x axis.



$\bar{y} = 0$ (5)

$$\bar{x} = \frac{\int_0^a \pi\rho x\,(a^2 - x^2)\,dx}{\int_0^a \pi\rho\,(a^2 - x^2)\,dx}$$ (10) (5)

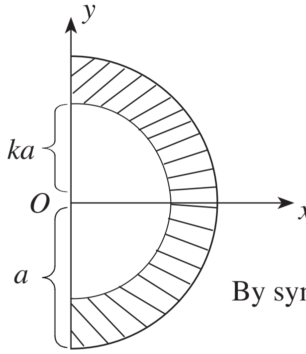
$$= \frac{\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a}{\left[a^2x - \frac{x^3}{3}\right]_0^a}$$ (5)

$$= \frac{3a}{8}$$ (5)

$\therefore \; G \equiv \left(\frac{3a}{8}, \; 0\right)$

30

(a)



By symmetry, the centre of mass lies on Ox.

Object	Mass	Distance from O to the centre of mass
Hemisphere	$\frac{2}{3}\pi a^3 \rho$	$\frac{3a}{8}$
Hemisphere which is removed	$\frac{2}{3}\pi (ka)^3 \rho$	$\frac{3ka}{8}$
Remaining portion	$\frac{2}{3}\pi a^3 \rho (1-k^3)$	$\overline{x}$

(10) (5)

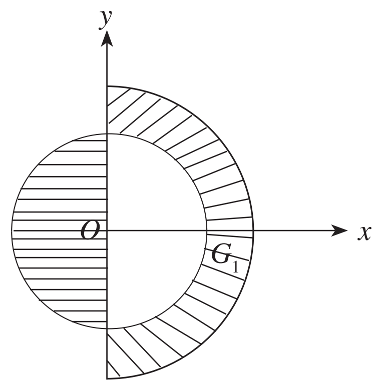
$$\overline{x} = \frac{\frac{2}{3}\pi a^3 \rho \frac{3a}{8} - \frac{2}{3}\pi k^3 a^3 \rho \frac{3ka}{8}}{\frac{2}{3}\pi a^3 \rho (1-k^3)} \quad (15)$$

$$= \frac{\frac{3a}{8}(1-k^4)}{(1-k^3)} = \frac{3a}{8}\frac{(1+k^2)(1-k)(1+k)}{(1-k)\ (1+k+k^2)}$$

$$= \frac{3a}{8}\frac{(1+k^2)\ (1+k)}{(1+k+k^2)} \quad (10)$$

40

(b) Let the centre of mass be $G_1(\overline{x}_1, \overline{y}_1)$
When the hemisphere that is removed is attached to the remaining portion as shown in the figure, due to symmetry about Ox we have $\overline{y}_1 = 0$. (5)



(i) Let the mass of the hemisphere that was removed be m, and the mass of the hemisphereof radius a be M.

$$\frac{m}{M} = \frac{\frac{2\pi}{3}k^3 a^3 \rho}{\frac{2\pi}{3}a^3 \rho} = k^3 \quad (5)$$

(5) (5)

$m = Mk^3$

20

(ii) The distance from O to the centre of mass of the composite object is $\bar{x}_1$.

$$\bar{x}_1 = \frac{(M-m)\,\bar{x} + m\left(-\frac{3}{8}ka\right)}{(M-m)+m} \quad (15)$$

Since $(M-m)\ \bar{x} = M\left(\frac{3a}{8}\right) - m\left(\frac{3}{8}ka\right)$, (5)

$$\bar{x}_1 = \frac{M\left(\frac{3a}{8}\right) - m\left(\frac{3}{8}ka\right) - m\left(\frac{3}{8}ka\right)}{M}$$

$$= \frac{3a}{8}\ \frac{(M-2mk)}{M}$$

$$= \frac{3a}{8}\left(1 - \frac{2m}{M}k\right) \quad (10)$$

The distance from O to the centre of mass of the composite object $= \frac{3a}{8}\,(1-2k^4)$

30

(iii) The centre of mass G_1 should coincide (5)

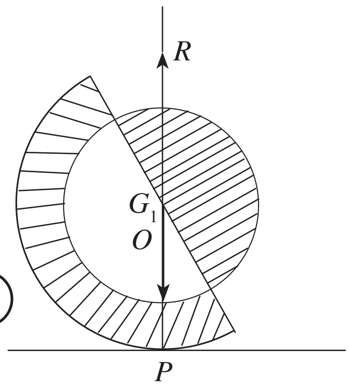
with O. That is, $\bar{x}_1 = 0$. (5)

$\frac{3a}{8}(1-2k^4) = 0$ (5)

$2k^4 = 1$

$k^2 = \pm\ \frac{1}{\sqrt{2}}$ (5)

Since $k^2 > 0$, $k^2 = \frac{1}{\sqrt{2}}$



(10)

30

17. (a) $P(A) = 0.1$, $P(A \cup B) = 0.37$ and $P(C) = 0.2$

(i) $P(A \cup B) = P(A) + P(B) - P(A)\,P(B)$ (5)

(Since A and B are independent) (5)

$0.37 = 0.1 + P(B) - 0.1\,P(B)$

$0.37 - 0.1 = 0.9\,P(B)$ (5)

$0.3 = P(B)$

15

(ii) $P(B' \mid A') = \frac{P(B' \cap A')}{P(A')}$ (5)

Here $P(B' \cap A') = P[(B \cup A)'] = 1 - P(A \cup B)$ (5)

$= 1 - 0.37 = 0.63$

$P(A') = 1 - P(A) = 1 - 0.1 = 0.9$ (5)

$\therefore\ P(B' \mid A') = \frac{0.63}{0.9} = 0.7$ (5)

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[See page 19

(iii) $P(A' \cap B' \cap C) = P(A')\, P(B')\, P(C)$ (5)

$= 0.9 \times 0.7 \times 0.2$

$= 0.126$ (5)

10

(iv) $X : (A \cap B' \cap C') \cup (A' \cap B \cap C') \cup (A' \cap B' \cap C)$ (5)

$P(X) = P(A \cap B' \cap C') + P(A' \cap B \cap C') + P(A' \cap B' \cap C)$

$= P(A)\ P(B')\, P(C') + P(A')\ P(B)\, P(C') + P(A')\ P(B')P(C)$ (5)

$= 0.1 \times 0.7 \times 0.8 + 0.9 \times 0.3 \times 0.8 + 0.9 \times 0.7 \times 0.2$ (10)

$= 0.398$

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$\Rightarrow P(A \mid X) = \dfrac{P\,(A \cap X)}{P\,(X)}$

$= \dfrac{P(A \cap B' \cap C')}{P\,(X)}$ (5)

$= \dfrac{0.1 \times 0.7 \times 0.8}{0.398}$

$= \dfrac{56}{398}$ (5)

$= \dfrac{28}{199}$

10

(b) (i) (α) Mean 8 $= \dfrac{\sum_{r=1}^{n} x_r}{n}$ (5)

Mean value of the marks $\bar{x} = \dfrac{28 + 56 + 23 + 94 + 8 + 5 + 13 + 846}{28}$ (5)

$= \dfrac{1073}{28}$

$= 38.32$ (5)

15

(β) Since the number 94 should have been 49;

Thus the drop of mark = 45

Since the number 05 should have been 50; (5)

Thus the rise of mark = 45

Therefore no change in the mean value. (5)

10

[*See page* 20

Standard deviation $= \sqrt{\dfrac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}}$ (5)

Variance $s_x^2 = \dfrac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}$

$= \dfrac{\sum_{i=1}^{n}(x_i^2 - 2x_i\bar{x} + \bar{x}^2)}{n}$ (5)

$= \dfrac{\sum_{r=1}^{n} x_i^2}{n} - 2\bar{x}\dfrac{\Sigma x_i}{n} + \bar{x}^2$

$= \dfrac{\sum_{r=1}^{n} x_i^2}{n} - 2\bar{x}^2 + \bar{x}^2$ (5)

$= \dfrac{\sum_{i=1}^{n} x_i^2}{n} - \bar{x}^2$

Let $X = \{x_1, x_2, ..., x_{20}\}$ and $Y = \{y_1, y_2, ..., y_{10}\}$.

Since $\sum_{i=1}^{20} x_i = 320$, $\sum_{i=1}^{20} x_i^2 = 5840$ (5)

and

$\sum_{i=1}^{10} y_i = 130$, $\sum_{i=1}^{10} y_i^2 = 2380$ (5)

$\therefore \ \bar{x} = \dfrac{\sum_{i=1}^{20} x_i}{20} = \dfrac{320}{20} = 16$ (5)

and $s_x^2 = \dfrac{\sum_{i=1}^{20} x_i^2}{20} - 16^2 = \dfrac{5840}{20} - 16^2$

$= 292 - 256 = 36$

$\therefore \ s_x = 6$ (5)

$\bar{y} = \dfrac{\sum_{i=1}^{10} y_i}{10} = \dfrac{130}{10} = 13$ (5)

$s_y^2 = \dfrac{\sum_{i=1}^{10} y_i^2}{100} - 13^2 = \dfrac{2380}{10} - 169 = 69$

$\therefore \ s_y = 8.30$ (5)

Let $Z = X \cup Y$.

$\bar{z} = \dfrac{\sum_{i=1}^{20} x_i + \sum_{i=1}^{10} y_i}{30}$

$= \dfrac{320 + 130}{30} = 15$ (5)

$$s_z^{\,2} = \frac{\sum_{i=1}^{20} x_i^2 + \sum_{i=1}^{10} y_i^2}{30} - \overline{z}^{\,2}$$ (5)

$= 274 - 225 \quad = \quad 49$

$s_z = 7$ (5)

60