ශ්‍රී ලංකා විභාග දෙපාර්තමේන්තුව
இலங்கைப் பரீட்சைத் திணைக்களம்
Department of Examinations, Sri Lanka

අධ්‍යයන පොදු සහතික පත්‍ර (උසස් පෙළ) විභාගය, 2016 අගෝස්තු
கல்விப் பொதுத் தராதரப் பத்திர (உயர் தர)ப் பரீட்சை, 2016 ஓகஸ்ற்
General Certificate of Education (Adv. Level) Examination, August 2016

3750

සංයුක්ත ගණිතය I
இணைந்த கணிதம் I
**Combined Mathematics I**

**10 E I**

පැය තුනයි
மூன்று மணித்தியாலம்
***Three hours***

| **Index Number** | | | | | | | |
|---|---|---|---|---|---|---|---|

## Instructions:

* *This question paper consists of two parts;* **Part A** (Questions 1 - 10) and **Part B** (Questions 11 - 17).
* **Part A:** *Answer **all** questions. Write your answers to each question in the space provided. You may use additional sheets if more space is needed.*
* **Part B:** *Answer **five** questions only. Write your answers on the sheets provided.*
* *At the end of the time allotted, tie the answer scripts of the two parts together so that* **Part A** *is on top of* **Part B** *and hand them over to the supervisor.*
* *You are permitted to remove **only*** **Part B** *of the question paper from the Examination Hall.*

**For Examiners' Use only**

| (10) Combined Mathematics I | | |
|---|---|---|
| **Part** | **Question No.** | **Marks** |
| A | 1 | |
| | 2 | |
| | 3 | |
| | 4 | |
| | 5 | |
| | 6 | |
| | 7 | |
| | 8 | |
| | 9 | |
| | 10 | |
| B | 11 | |
| | 12 | |
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| | 16 | |
| | 17 | |
| | **Total** | |
| | **Percentage** | |

| | |
|---|---|
| Paper I | |
| Paper II | |
| Total | |
| Final Marks | |

**Final Marks**

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**Code Numbers**

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| Marking Examiner | | |
| Checked by: | 1 | |
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| Supervised by: | | |



[see page two

## Part A

1. Using the **Principle of Mathematical Induction**, prove that $\sum_{r=1}^{n} r(r+1) = \frac{n}{3}(n+1)(n+2)$ for all $n \in \mathbb{Z}^{+}$.

2. Sketch the graphs of $y = |x| + 1$ and $y = 2|x-1|$ in the same diagram. **Hence or otherwise**, find all real values of $x$ satisfying the inequality $|x| + 1 > 2|x-1|$.



[see page three

**3.** Sketch on the same Argand diagram, the loci of points representing complex numbers $z$ satisfying

(i) $|z - i| = 1$, (ii) $\text{Arg}\,(z - i) = \frac{\pi}{6}$

and find the complex number represented by the point of intersection of these loci in the form $r\,(\cos\theta + i\sin\theta)$, where $r > 0$ and $0 < \theta < \frac{\pi}{2}$.

**4.** How many different numbers with five digits can be made from the digits 1, 2, 3, 4 and 5, if each digit is used only once?

How many of these numbers

(i) are even numbers?

(ii) have the digits 3 and 4 next to each other?



[see page four

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**5.** Let $\alpha > 0$. Find the value of $\alpha$ such that $\lim_{x \to 0} \dfrac{1-\cos(\alpha x)}{\sqrt{4+x^2}-\sqrt{4-x^2}} = 16$.

**6.** Show that the area of the region enclosed by the curves $y = x^2$ and $y = 2x - x^2$ is $\frac{1}{3}$ square units.



[see page five

**7.** A curve $C$ is given by the parametric equations $x = 3\sin^2\frac{\theta}{2}$, $y = \sin^3\theta$ for $0 < \theta < \frac{\pi}{4}$. Show that $\frac{dy}{dx} = \sin 2\theta$.

If the gradient of the tangent at a point $P$ on $C$ is $\frac{\sqrt{3}}{2}$, find the value of the parameter $\theta$ corresponding to $P$.

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**8.** Let $l$ be the straight line that passes through the origin and the point of intersection of the straight lines $2x + 3y - k = 0$ and $x - y + 1 = 0$, where $k$ $(\neq 0)$ is a constant. Find the equation of $l$ in terms of $k$.

It is given that the two points $(1, 1)$ and $(3, 4)$ are on the same side of $l$. Show that $k < 18$.

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[see page six

**9.** Let $A \equiv (1, 2)$, $B \equiv (-5, 4)$ and $S$ be the circle with $AB$ as a diameter. Find the equations of

(i) the circle $S$, and

(ii) the circle with centre $(1, 1)$ which intersects $S$ orthogonally.

**10.** Solve the equation $\cos x + \cos 2x + \cos 3x = \sin x + \sin 2x + \sin 3x$ for $0 \leq x \leq \frac{\pi}{2}$.

General Certificate of Education (Adv. Level) Examination, August 2016

Combined Mathematics I

10 E I

**PART B**

* Answer **five** questions only.

**11.** (*a*) Let $a, b, c \in \mathbb{R}$ such that $a \neq 0$ and $a + b + c \neq 0$, and let $f(x) = ax^2 + bx + c$.

Show that 1 **is not** a root of the equation $f(x) = 0$.

Let $\alpha$ and $\beta$ be the roots of $f(x) = 0$.

Show that $(\alpha - 1)(\beta - 1) = \frac{1}{a}(a + b + c)$ and that the quadratic equation with $\frac{1}{\alpha - 1}$ and $\frac{1}{\beta - 1}$ as the roots is given by $g(x) = 0$, where $g(x) = (a + b + c)\,x^2 + (2a + b)\,x + a$.

Now, let $a > 0$ and $a + b + c > 0$.

Show that the minimum value $m_1$ of $f(x)$ is given by $m_1 = -\frac{\Delta}{4a}$, where $\Delta = b^2 - 4ac$.
Let $m_2$ be the minimum value of $g(x)$. Deduce that $(a + b + c)\,m_2 = am_1$.

**Hence**, show that $f(x) \geq 0$ for all $x \in \mathbb{R}$ if and only if $g(x) \geq 0$ for all $x \in \mathbb{R}$.

(*b*) Let $p(x) = x^3 + 2x^2 + 3x - 1$ and $q(x) = x^2 + 3x + 6$. Using the remainder theorem, find the remainder when $p(x)$ is divided by $(x - 1)$ and the remainder when $q(x)$ is divided by $(x - 2)$.

Verify that $p(x) = (x - 1)\,q(x) + 5$, and find the remainder when $p(x)$ is divided by $(x - 1)(x - 2)$.

**12.** (*a*) Let $n \in \mathbb{Z}^+$. State, in the usual notation, the binomial expansion for $(1 + x)^n$.

Show, in the usual notation, that $\frac{{}^nC_{r+1}}{{}^nC_r} = \frac{n - r}{r + 1}$ for $r = 0, 1, 2, \ldots, n - 1$.

The coefficients of $x^r$, $x^{r+1}$ and $x^{r+2}$ taken in that order, in the binomial expansion of $(1 + x)^n$ are in the ratios $1 : 2 : 3$. In this case, show that $n = 14$ and $r = 4$.

(*b*) Let $U_r = \frac{10r + 9}{(2r - 3)(2r - 1)(2r + 1)}$ and $f(r) = r(Ar + B)$ for $r \in \mathbb{Z}^+$, where $A$ and $B$ are real constants.

Find the values of constants $A$ and $B$ such that

$$U_r = \frac{f(r)}{(2r - 3)(2r - 1)} - \frac{f(r + 1)}{(2r - 1)(2r + 1)} \text{ for } r \in \mathbb{Z}^+.$$

Show that $\sum_{r=1}^{n} U_r = -3 - \frac{(n + 1)(2n + 3)}{(4n^2 - 1)}$ for $n \in \mathbb{Z}^+$.

Show further that the infinite series $\sum_{r=1}^{\infty} U_r$ is convergent and find its sum.



[see page eight

13.(a) Let $\mathbf{A} = \begin{pmatrix} -4 & -6 \\ 3 & 5 \end{pmatrix}$, $\mathbf{X} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\mathbf{Y} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

Find real constants $\lambda$ and $\mu$ such that $\mathbf{AX} = \lambda\mathbf{X}$ and $\mathbf{AY} = \mu\mathbf{Y}$.

Let $\mathbf{P} = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$. Find $\mathbf{P}^{-1}$ and $\mathbf{AP}$, and show that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$, where $\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$.

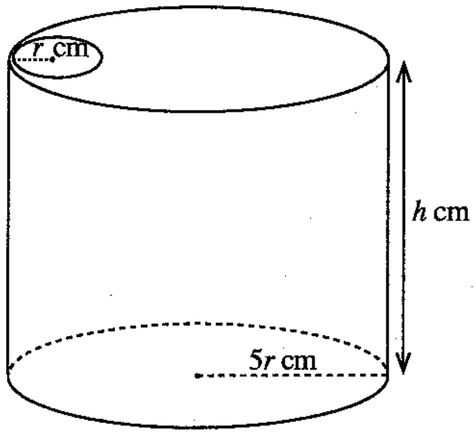
(b) In an Argand diagram, the point $A$ represents the complex number $2+i$. The point $B$ is such that $OB = 2(OA)$ and $A\hat{O}B = \frac{\pi}{4}$, where $O$ is the origin and $A\hat{O}B$ is measured counter-clockwise from $OA$. Find the complex number represented by the point $B$.

Also, find the complex number represented by the point $C$ such that $OACB$ is a parallelogram.

(c) Let $z \in \mathbb{C}$ and $w = \dfrac{2}{1+i} + \dfrac{5z}{2+i}$. It is given that $\operatorname{Im} w = -1$ and $|w - 1 + i| = 5$.
Show that $z = \pm(2 + i)$.

14.(a) Let $f(x) = \dfrac{(x-3)^2}{x^2-1}$ for $x \neq \pm 1$.
Show that $f'(x)$, the derivative of $f(x)$, is given by $f'(x) = \dfrac{2(x-3)(3x-1)}{(x^2-1)^2}$.
Write down the equations of the asymptotes of $y = f(x)$.
Find the coordinates of the point at which the horizontal asymptote intersects the curve $y = f(x)$.
Sketch the graph of $y = f(x)$ indicating the asymptotes and the turning points.

(b) A thin metal container, in the shape of a right circular cylinder of radius $5r$ cm and height $h$ cm has a circular lid of radius $5r$ cm with a circular hole of radius $r$ cm. (See the figure.) The volume of the container is given to be $245\pi$ cm$^3$. Show that the surface area $S$ cm$^2$ of the container with the lid containing the hole is given by $S = 49\pi\left(r^2 + \dfrac{2}{r}\right)$ for $r > 0$.
Find the value of $r$ such that $S$ is minimum.




15.(a) (i) Find $\displaystyle\int \frac{dx}{\sqrt{3+2x-x^2}}$.

(ii) Find $\dfrac{d}{dx}\left(\sqrt{3+2x-x^2}\right)$ and **hence**, find $\displaystyle\int \frac{x-1}{\sqrt{3+2x-x^2}}\,dx$.

Using the above integrals, find $\displaystyle\int \frac{x+1}{\sqrt{3+2x-x^2}}\,dx$.

(b) Express $\dfrac{2x-1}{(x+1)(x^2+1)}$ in partial fractions and **hence**, find $\displaystyle\int \frac{(2x-1)}{(x+1)(x^2+1)}\,dx$.

(c) (i) Let $n \neq -1$. Using integration by parts, find $\int x^n(\ln x)\,dx$.

(ii) Evaluate $\displaystyle\int_1^3 \frac{\ln x}{x}\,dx$.



[see page nine

**16.** (*a*) The equation of the diagonal $AC$ of a rhombus $ABCD$ is $3x - y = 3$ and $B \equiv (3, 1)$. Also, the equation of $CD$ is $x + ky = 4$, where $k$ is a real constant. Find the value of $k$ and the equation of $BC$.

(*b*) Sketch the circles, $C_1$ and $C_2$ given by the equations $x^2 + y^2 = 4$ and $(x-1)^2 + y^2 = 1$ respectively, indicating clearly their point of contact.

A circle $C_3$ touches $C_1$ internally and $C_2$ externally. Show that the centre of $C_3$ lies on the curve $8x^2 + 9y^2 - 8x - 16 = 0$.

**17.** (*a*) Write down the trigonometric identity for $\tan(\alpha + \beta)$ in terms of $\tan\alpha$ and $\tan\beta$.

**Hence**, obtain $\tan 2\theta$ in terms of $\tan\theta$, and show that $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.

By substituting $\theta = \dfrac{5\pi}{12}$ in the last equation, verify that $\tan\dfrac{5\pi}{12}$ is a solution of $x^3 - 3x^2 - 3x + 1 = 0$.

Given further that $x^3 - 3x^2 - 3x + 1 = (x + 1)(x^2 - 4x + 1)$, deduce that $\tan\dfrac{5\pi}{12} = 2 + \sqrt{3}$.

(*b*) Show that $\tan^2\dfrac{A}{2} = \dfrac{1 - \cos A}{1 + \cos A}$ for $0 < A < \pi$.

In the usual notation, using the Cosine Rule for a triangle $ABC$, show that

$$(a + b + c)(b + c - a)\tan^2\frac{A}{2} = (a + b - c)(a + c - b).$$

(*c*) Show that $\sin^{-1}\left(\dfrac{3}{5}\right) + \sin^{-1}\left(\dfrac{5}{13}\right) = \sin^{-1}\left(\dfrac{56}{65}\right)$.

* * *

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இலங்கைப் பரீட்சைத் திணைக்களம் இலங்கைப் பரீட்சைத் திணைக்களம்
Department of Examinations, Sri Lanka Department of Examinations, Sri Lanka

ශ්‍රී ලංකා විභාග දෙපාර්තමේන්තුව
இலங்கைப் பரீட்சைத் திணைக்களம்
Department of Examinations, Sri Lanka

අධ්‍යයන පොදු සහතික පත්‍ර (උසස් පෙළ) විභාගය, 2016 අගෝස්තු
கல்விப் பொதுத் தராதரப் பத்திர (உயர் தர)ப் பரீட்சை, 2016 ஓகஸ்ற்
General Certificate of Education (Adv. Level) Examination, August 2016

සංයුක්ත ගණිතය II
இணைந்த கணிதம் II
**Combined Mathematics II**

**10 E II**

පැය තුනයි
மூன்று மணித்தியாலம்
***Three hours***

| **Index Number** | | | | | | | |
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## Instructions:

* *This question paper consists of two parts;*
**Part A** (Questions 1 – 10) *and* **Part B** (Questions 11 – 17)

* **Part A:**
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* **Part B:**
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* *At the end of the time allotted, tie the answer scripts of the two parts together so that* **Part A** *is on top of* **Part B** *and hand them over to the supervisor.*

* *You are permitted to remove **only** **Part B** of the question paper from the Examination Hall.*

* In this question paper, $g$ denotes the acceleration due to gravity.

### For Examiners' Use only

**(10) Combined Mathematics II**

| Part | Question No. | Marks |
|---|---|---|
| A | 1 | |
| | 2 | |
| | 3 | |
| | 4 | |
| | 5 | |
| | 6 | |
| | 7 | |
| | 8 | |
| | 9 | |
| | 10 | |
| B | 11 | |
| | 12 | |
| | 13 | |
| | 14 | |
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| | 17 | |
| | **Total** | |
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| Paper I | |
| Paper II | |
| Total | |
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**Final Marks**

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**Code Numbers**

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| Marking Examiner | |
| Checked by: 1 | |
| Checked by: 2 | |
| Supervised by: | |



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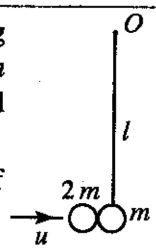
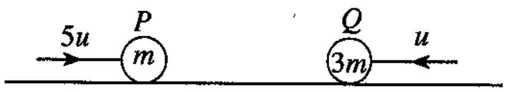
## Part A

1. A particle of mass $m$ hangs in equilibrium at one end of a light inextensible string of length $l$ whose other end is tied to a fixed point $O$. Another particle of mass $2m$ collides horizontally with velocity $u$ with the first particle and coalesces with it. Find the velocity with which the composite particle begins to move.

   Show that if $u = \sqrt{gl}$, then the composite particle reaches a maximum height of $\frac{2l}{9}$ above its initial level.



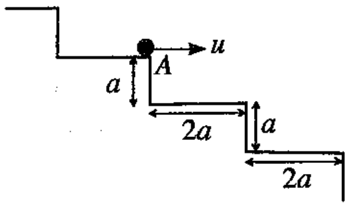

2. A particle $P$ of mass $m$ and a particle $Q$ of mass $3m$ move on a smooth horizontal table along the same straight line towards each other with speeds $5u$ and $u$ respectively, as shown in the figure. After their impact, $P$ and $Q$ move **away from each other** with speeds $u$ and $v$ respectively. Find $v$ in terms of $u$, and show that the coefficient of restitution between $P$ and $Q$ is $\frac{1}{3}$.






[see page three

3. A particle $P$, projected horizontally with velocity $u$ given by $u = \frac{3}{2}\sqrt{ga}$ from a point $A$ at the edge of a step of a fixed stairway perpendicular to that edge, moves under gravity. Each step is of height $a$ and length $2a$ (see the figure). Show that the particle $P$ will **not hit** the first step below $A$, and it will hit the second step below $A$ at a horizontal distance $3a$ from $A$.




4. A car of mass $M$ kg moves along a straight level road against a resistance of constant magnitude $R$ N. At an instant when the car is moving at speed $v$ m s$^{-1}$, its acceleration is $a$ m s$^{-2}$. Show that the power of its engine at this instant is $(R + Ma)v$ W.

The car then moves with a constant speed $v_1$ m s$^{-1}$ against a resistance of the same constant magnitude $R$ N up a straight road inclined at an angle $\alpha$ to the horizontal, working at the same power. Show that $v_1 = \dfrac{(R + Ma)v}{R + Mg \sin \alpha}$.



[see page four

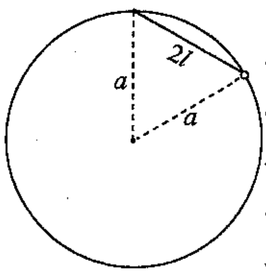
5. In the usual notation, let $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$, $\mathbf{b} = 4\mathbf{i} + 3\mathbf{j}$ and $\mathbf{c} = \alpha\mathbf{i} + (1 - \alpha)\,\mathbf{j}$, where $\alpha \in \mathbb{R}$.

Find (i) $|\mathbf{a}|$ and $|\mathbf{b}|$,

(ii) $\mathbf{a}\cdot\mathbf{c}$ and $\mathbf{b}\cdot\mathbf{c}$ in terms of $\alpha$.

If the angle between **a** and **c** is equal to the angle between **b** and **c**, show that $\alpha = \frac{1}{2}$.

6. One end of a light inextensible string of length $2l$ is attached to the highest point of a thin smooth rigid circular wire of radius $a\left(> \sqrt{2}\,l\right)$ which is fixed in a vertical plane. A small smooth bead of weight $w$, which is free to move along the wire, is attached to the other end of the string. The bead is in equilibrium with the string taut, as in the figure. Mark the forces acting on the bead and show that the tension of the string is $\frac{2wl}{a}$.






[see page five

7. Let $A$ and $B$ be two events of a sample space $\Omega$. In the usual notation, $P(A) = p$, $P(B) = \frac{p}{2}$ and $P(A \cup B) - P(A \cap B) = \frac{2p}{3}$, where $p > 0$. Find $P(A \cap B)$ in terms of $p$.

Deduce that if $A$ and $B$ are independent events, then $p = \frac{5}{6}$.

8. A bag contains 6 white balls and $n$ black balls which are equal in all respects, except for colour. Two balls are taken out at random from the bag, one after the other, without replacement. The probability that the first ball is white and the second ball is black is $\frac{4}{15}$. Find the value of $n$.



[see page six

**9.** The mean of three distinct integers less than 11 is 7. When two more integers are taken, the mean of all five integers is 5. Also, the only mode of these five integers is 3. Find the five integers.

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**10.** An arrow is shot at a rotating circular target-board consisting of five equal sectors numbered 1, 2, 3, 4 and 5. The number of times the arrow hits each of the sectors is given in the following frequency table, where $p$ and $q$ are constants.

| Number | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Frequency | 1 | $p$ | $q$ | 5 | 2 |

If the mean and the variance of the above data are given to be 3 and $\frac{6}{5}$ respectively, find the values of $p$ and $q$.

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கல்விப் பொதுத் தராதரப் பத்திர (உயர் தரப்) பரீட்சை, 2016 ஓகஸ்ற்
General Certificate of Education (Adv. Level) Examination, August 2016

සංයුක්ත ගණිතය II
இணைந்த கணிதம் II
**Combined Mathematics II**

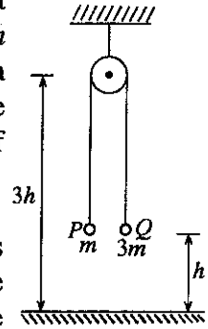
**10 E II**

## PART B

* Answer **five** questions only.

(In this question paper, $g$ denotes the acceleration due to gravity.)

**11.** (*a*) A particle $P$ of mass $m$ is connected to a particle $Q$ of mass $3m$ by a light inextensible string passing over a small smooth pulley fixed at a height $3h$ above an inelastic horizontal floor. Initially the two particles are held at a height $h$ above the floor with the string taut, and released from rest. (See the adjoining figure.) Applying Newton's second law separately to the motions of $P$ and $Q$, show that the magnitude of acceleration of each particle is $\frac{g}{2}$.

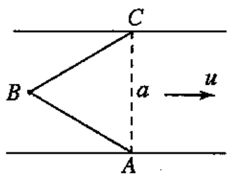



After a time $t_0$ the particle $Q$ strikes the floor, comes to rest instantly, remains at rest for a further time $t_1$ and begins to move up. Sketch the velocity-time graphs separately for the motions of the two particles $P$ and $Q$ until the particle $Q$ begins to move up.

Using these graphs, show that $t_0 = 2\sqrt{\frac{h}{g}}$ and find $t_1$ in terms of $g$ and $h$.

Show further that the particle $P$ reaches a maximum height $\frac{5h}{2}$ above the floor.

(*b*) A straight river of breadth $a$ flows with uniform speed $u$. The points $A$ and $C$ are situated on opposite banks of the river such that the line $AC$ is perpendicular to the direction of flow of the river. Also, a stationary buoy $B$ is fixed in the middle of the river, on the upstream side of $AC$ such that $ABC$ is an equilateral triangle. (See the adjoining figure.)

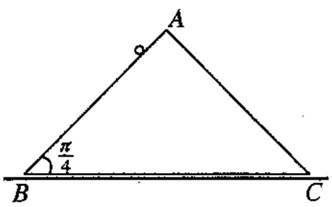



A boat moving with speed $v$ ( $> u$) relative to water starts off from $A$ and moves until it reaches $B$. Then it moves from $B$ to $C$. Sketch the velocity triangles for the motions of the boat from $A$ to $B$ and from $B$ to $C$.

Show that the speed of the boat in its motion from $A$ to $B$ is $\frac{1}{2}\left(\sqrt{4v^2 - u^2} - \sqrt{3}u\right)$ and find its speed in the motion from $B$ to $C$.

**Hence**, show that the total time taken by the boat for the paths $AB$ and $BC$ is $\frac{a\sqrt{4v^2 - u^2}}{v^2 - u^2}$.

**12.** (*a*) The triangle $ABC$ in the figure is a vertical cross-section through the centre of gravity of a uniform wedge of mass $2m$. The line $AB$ is a line of greatest slope of the face containing it and $A\hat{B}C = \frac{\pi}{4}$. The wedge is placed with the face containing $BC$ on a rough horizontal floor. The face containing $AB$ is smooth. A particle of mass $m$ is held on $AB$ as in the figure and the system is released from rest. It is given that the wedge moves in the direction of $\overrightarrow{BC}$ and that the magnitude of the frictional force exerted on the wedge by the floor is $\frac{R}{6}$, where $R$ is the magnitude of the normal reaction exerted on the wedge by the floor. Obtain equations which are sufficient to determine $R$, in terms of $m$ and $g$.



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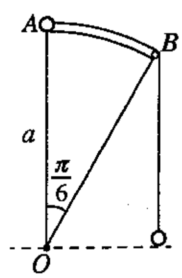
(*b*) *OAB* in the figure is a circular sector of radius $a$ subtending an angle $\frac{\pi}{6}$ at the centre $O$ with $OA$ vertical. It is a cross-section perpendicular to the axis of a smooth cylindrical sector fixed with its axis horizontal. One end of a light inextensible string passing over a small smooth pulley fixed at $B$ is attached to a particle $P$ of mass $3m$ and the other end is attached to a particle $Q$ of mass $m$. Initially, the particle $P$ is held at $A$ and the particle $Q$ hangs freely at the horizontal level of $O$. The system is released from rest in this position, with the string taut. When $OP$ makes an angle $\theta\left(0 < \theta < \frac{\pi}{6}\right)$ with the upward vertical, show that $2a\dot{\theta}^2 = 3g(1-\cos\theta) + g\theta$ and that the tension in the string is $\frac{3}{4}mg\,(1-\sin\theta)$, and find the normal reaction on the particle $P$.




**13.** One end of a light elastic string of natural length $a$ and modulus of elasticity $4mg$ is tied to a fixed point $O$ and the other end to a particle $P$ of mass $m$. The particle $P$ is released from rest at $O$. Find the velocity of the particle $P$ when it passes through the point $A$, where $OA = a$.

Show that the length of the string $x(\geq a)$ satisfies the equation $\ddot{x} + \frac{4g}{a}\left(x - \frac{5a}{4}\right) = 0$.

Taking $X = x - \frac{5a}{4}$, express the above equation in the form $\ddot{X} + \omega^2 X = 0$, where $\omega(>0)$ is a constant to be determined.

Assuming that $\dot{X}^2 = \omega^2\left(c^2 - X^2\right)$, find the amplitude $c$ of this simple harmonic motion.

Let $L$ be the lowest point reached by the particle $P$. Show that the time taken by $P$ to move from $A$ to $L$ is $\frac{1}{2}\sqrt{\frac{a}{g}}\left\{\pi - \cos^{-1}\left(\frac{1}{3}\right)\right\}$.

At the instant when the particle $P$ is at $L$, another particle of mass $\lambda m$ $(1 \leq \lambda < 3)$ is gently attached to $P$. Show that the equation of motion of the composite particle of mass $(1+\lambda)m$ is $\ddot{x} + \frac{4g}{(1+\lambda)a}\left\{x - (5+\lambda)\frac{a}{4}\right\} = 0$.

Show further that the composite particle performs complete simple harmonic motion with amplitude $(3-\lambda)\frac{a}{4}$.

**14.** (*a*) The position vectors of two points $A$ and $B$ with respect to an origin $O$ are **a** and **b** respectively, where $O$, $A$ and $B$ are **not collinear**. Let $C$ be the point such that $\overrightarrow{OC} = \frac{1}{3}\overrightarrow{OB}$ and let $D$ be the point such that $\overrightarrow{OD} = \frac{1}{2}\overrightarrow{AB}$. By expressing $\overrightarrow{AC}$ and $\overrightarrow{AD}$ in terms of **a** and **b**, show that $\overrightarrow{AD} = \frac{3}{2}\overrightarrow{AC}$.

Let $P$ and $Q$ be the points on $AB$ and $OD$ respectively, such that $\overrightarrow{AP} = \lambda\overrightarrow{AB}$ and $\overrightarrow{OQ} = (1-\lambda)\overrightarrow{OD}$, where $0 < \lambda < 1$. Show that $\overrightarrow{PC} = 2\overrightarrow{CQ}$.

(*b*) In a parallelogram $ABCD$, let $AB = 2$ m and $AD = 1$ m, and let $B\hat{A}D = \frac{\pi}{3}$. Also, let $E$ be the mid-point of $CD$. Forces of magnitudes 5, 5, 2, 4 and 3 newtons act along $AB$, $BC$, $DC$, $DA$ and $BE$ respectively, in the directions indicated by order of the letters. Show that their resultant force is parallel to $\overrightarrow{AE}$, and find its magnitude.

Also, show that the line of action of the resultant force meets $AB$ produced at a distance $\frac{3}{2}$ m from $B$.

An additional force acting through $C$ is now added to the above system of forces so that the resultant force of the new system is along $AE$. Find the magnitude and direction of the additional force.



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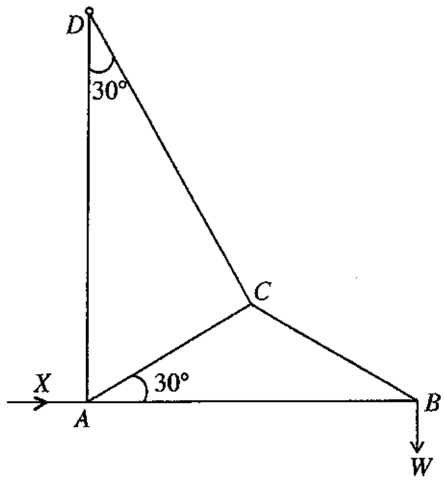
**15.** (*a*) Four equal uniform rods, each of weight $w_1$, are smoothly jointed at their ends to form a rhombus *ABCD*. The mid-points of *BC* and *CD* are connected by a light rod such that $B\hat{A}D = 2\theta$. Each of the joints *B* and *D* carries equal loads of weight $w_2$. The system, hanging symmetrically from the joint *A*, is in equilibrium in a vertical plane with the light rod horizontal. Show that the thrust in the light rod is $2(2w_1 + w_2)\tan\theta$.

(*b*) The adjoining figure represents a framework of five light rods *AB*, *BC*, *CD*, *AC* and *AD*, smoothly jointed at the ends. It is given that $AC = CB$ and $B\hat{A}C = 30° = A\hat{D}C$. The framework is smoothly hinged at *D*. A weight *W* is suspended at the joint *B* and the framework is kept in equilibrium in a vertical plane with *AB* horizontal and *AD* vertical, by a horizontal force of magnitude *X* acting at *A*.

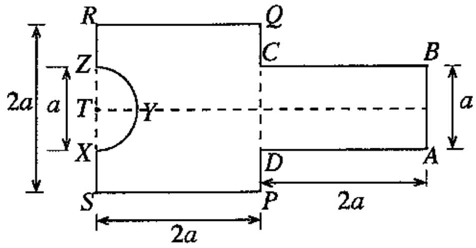
Using Bow's notation, draw stress diagrams for the joints *B*, *C* and *A* in the same figure.

**Hence**, find the value of *X* and the stresses in all rods, distinguishing between tensions and thrusts.

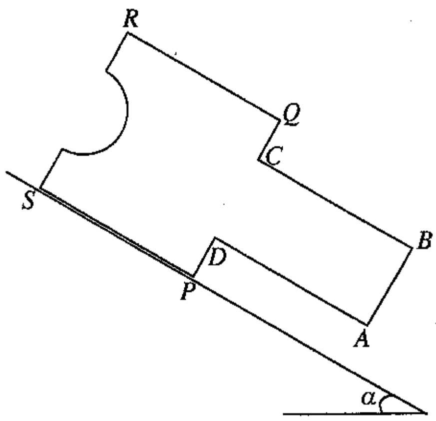



**16.** Show that the centre of mass of a uniform semi-circular lamina of radius $r$ and centre $O$ is at a distance $\frac{4r}{3\pi}$ from $O$.

As shown in the adjoining figure, a uniform plane lamina *L* is made by rigidly attaching a rectangle *ABCD* to a square *PQRS* such that *DC* and *PQ* lie on the same line with their mid-points coinciding, and removing a semi-circular region *XYZ* of radius $\frac{a}{2}$ centred at the mid-point *T* of *RS*. It is given that $AB = a$ and $AD = PQ = 2a$. Show that the centre of mass of the lamina *L* lies on the axis of symmetry at a distance $ka$ from *RS*, where $k = \frac{238}{3(48-\pi)}$.




As shown in the adjoining figure, the lamina *L* is in equilibrium on a rough plane inclined at an angle $\alpha$ to the horizontal with its plane vertical and the edge *PS* on a line of greatest slope such that the point *P* lies below *S*. Show that $\tan\alpha < (2-k)$ and $\mu \geq \tan\alpha$, where $\mu$ is the coefficient of friction between the lamina and the inclined plane.



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17. (*a*) An unbiased cubical die $A$ shows 1, 2, 3, 3, 4, 5 on its six separate faces. The die $A$ is tossed twice. Find the probability that the sum of the two numbers obtained is 6.

Another die $B$, identical to $A$ in all respects except for the numbers on the faces, shows 2, 2, 3, 4, 4, 5 on its six separate faces. The die $B$ is tossed twice. Find the probability that the sum of the two numbers obtained is 6.

Now, the two dice $A$ and $B$ are put in a box. One die is taken out of the box at random and tossed twice. Given that the sum of the two numbers obtained is 6, find the probability that the die taken out of the box is the die $A$.

(*b*) The mean and the standard deviation of $n$ numbers $x_1, x_2, \ldots, x_n$ are $\mu_1$ and $\sigma_1$ respectively, and the mean and the standard deviation of $m$ numbers $y_1, y_2, \ldots, y_m$ are $\mu_2$ and $\sigma_2$ respectively. Let the mean and the standard deviation of all of these $n + m$ numbers be $\mu_3$ and $\sigma_3$ respectively.

Show that $\mu_3 = \dfrac{n\mu_1 + m\mu_2}{n+m}$.

Let $d_1 = \mu_3 - \mu_1$. Show that $\sum_{i=1}^{n} (x_i - \mu_3)^2 = n(\sigma_1^2 + d_1^2)$.

By taking $d_2 = \mu_3 - \mu_2$, write down a similar expression for $\sum_{j=1}^{m} (y_j - \mu_3)^2$.

Deduce that $\sigma_3^2 = \dfrac{(n\sigma_1^2 + m\sigma_2^2) + (nd_1^2 + md_2^2)}{n+m}$.

The number of copies sold per day, during the first 100 days after publishing a new book, had mean 2.3 and variance 0.8. During the next 100 days, the number of copies sold per day had mean 1.7 and variance 0.5. Find the mean and the variance of the number of copies sold per day during the first 200 days.

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