Confidential

NEW/OLD

Department of Examination - Sri Lanka

G.C.E. (A/L) Examination - 2020

10 - Combined Mathematics - I

NEW/OLD Syllabus

Marking Scheme

This document has been prepared for the use of Marking Examiners. Some changes would be made according to the views presented at the Chief Examiners' meeting.

Amendments to be included

G.C.E. (A.L.) Examination - 2020

10 - Combined Mathematics I

(New/Old Syllabus)

Distribution of Marks

Paper I

Part A : 10 × 25 = 250

Part B : 05 × 150 = 750

Total = 1000 / 10

Paper I Final Mark = 100

Common Techniques of Marking Answer Scripts.

It is compulsory to adhere to the following standard method in marking answer scripts and entering marks into the mark sheets.

1. Use a red color ball point pen for marking. (Only Chief/Additional Chief Examiner may use a mauve color pen.)
2. Note down Examiner's Code Number and initials on the front page of each answer script.
3. Write off any numerals written wrong with a clear single line and authenticate the alterations with Examiner's initials.
4. Write down marks of each subsection in a △ and write the final marks of each question as a rational number in a □ with the question number. Use the column assigned for Examiners to write down marks.

Example: **Question No. 03**

(i) .... √ $\frac{4}{5}$ (in △)

(ii) .... √ $\frac{3}{5}$ (in △)

(iii) .... √ $\frac{3}{5}$ (in △)

(03) (i) $\frac{4}{5}$ + (ii) $\frac{3}{5}$ + (iii) $\frac{3}{5}$ = $\boxed{\frac{10}{15}}$

MCQ answer scripts: (Template)

1. Marking templets for G.C.E.(A/L) and GIT examination will be provided by the Department of Examinations itself. Marking examiners bear the responsibility of using correctly prepared and certified templates.
2. Then, check the answer scripts carefully. If there are more than one or no answers Marked to a certain question write off the options with a line. Sometimes candidates may have erased an option marked previously and selected another option. In such occasions, if the erasure is not clear write off those options too.
3. Place the template on the answer script correctly. Mark the right answers with a 'V' and the wrong answers with a 'X' against the options column. Write down the number of correct answers inside the cage given under each column. Then, add those numbers and write the number of correct answers in the relevant cage.

Structured essay type and assay type answer scripts:

1. Cross off any pages left blank by candidates. Underline wrong or unsuitable answers. Show areas where marks can be offered with check marks.

2. Use the right margin of the overland paper to write down the marks.

3. Write down the marks given for each question against the question number in the relevant cage on the front page in two digits. Selection of questions should be in accordance with the instructions given in the question paper. Mark all answers and transfer the marks to the front page, and write off answers with lower marks if extra questions have been answered against instructions.

4. Add the total carefully and write in the relevant cage on the front page. Turn pages of answer script and add all the marks given for all answers again. Check whether that total tallies with the total marks written on the front page.

Preparation of Mark Sheets.

Except for the subjects with a single question paper, final marks of two papers will not be calculated within the evaluation board this time. Therefore, add separate mark sheets for each of the question paper. Write paper 01 marks in the paper 01 column of the mark sheet and write them in words too. Write paper II Marks in the paper II Column and wright the relevant details. For the subject 51 Art, marks for Papers 01, 02 and 03 should be entered numerically in the mark sheets.

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New Syllabus

1. Using the **Principle of Mathematical Induction**, prove that $\sum_{r=1}^{n}(4r+1) = n(2n+3)$ for all $n \in \mathbb{Z}^{+}$.

For $n = 1$, L.H.S. = $4 + 1 = 5$ and R.H.S. = $1(2 + 3) = 5$ and hence, L.H.S. = R.H.S.

Hence the result is true for $n = 1$. (5)

Let k be any positive integer and suppose that the result in true for $n = k$.

i.e. $\sum_{r=1}^{k} (4r + 1) = k(2k + 3)$. (5)

Now $\sum_{r=1}^{k+1} (4r + 1) = \sum_{r=1}^{k} (4r + 1) + \{4(k + 1) + 1\}$

$= k(2k + 3) + (4k + 5)$ (5)

$= 2k^2 + 7k + 5$

$= (k + 1)(2k + 5)$ (5)

$= (k + 1)[2(k + 1) + 3]$

Hence, if the result is true for $n = k$, it is also true for $n = k + 1$. The result is true for $n = 1$ also.

Hence, by the Principle of Mathematical Induction, the result is true for all $n \in \mathbb{Z}^{+}$. (5)

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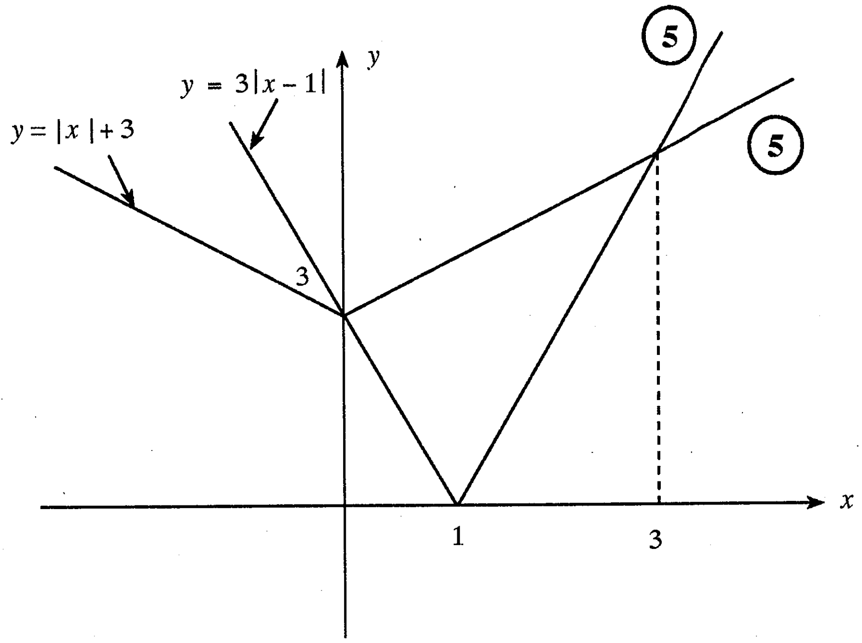
2. Sketch the graphs of $y = 3|x-1|$ and $y = |x|+3$ in the same diagram.
Hence or otherwise, find all real values of x satisfying the inequality $3|2x-1| > 2|x|+3$.



One point of intersection is given by $x = 0$. The other point of intersection is given by

$$3(x-1) = x+3 \text{ for } x > 1.$$

This gives $x = 3$. (5)

$3|2x-1| > 2|x| + 3$

$\Leftrightarrow$ $3|u-1| > |u| + 3$, where $u = 2x$. (5)

$\Leftrightarrow$ $u < 0$ or $u > 3$ (From the graphs)

$\Leftrightarrow$ $x < 0$ or $x > \frac{3}{2}$. (5)

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Aliter 1:

For the graphs (5) + (5), as before.

Aliter for the values of x:

$3\,|2x-1| > 2\,|x| + 3$

Case (i) $x \geq \frac{1}{2}$

Then, $3\,|2x-1| > 2\,|x| + 3 \Leftrightarrow 3(2x-1) > 2x+3$

$\Leftrightarrow 6x - 3 > 2x + 3$

$\Leftrightarrow x > \frac{3}{2}$

Hence, in this case, the solutions are the values of x satisfying $x > \frac{3}{2}$.

Case (ii) $0 \leq x < \frac{1}{2}$

Then, $3\,|2x-1| > 2\,|x| + 3 \Leftrightarrow -6x + 3 > 2x + 3$

$\Leftrightarrow 0 > 8x$

$\Leftrightarrow 0 > x$

Hence, in this case, there are no solutions.

All 3 cases with correct solutions	(10)
Any 2 cases with correct solutions	(5)

Case (iii) $x < 0$

Then, $3\,|2x-1| > 2\,|x| + 3 \Leftrightarrow -6x + 3 > -2x + 3$

$\Leftrightarrow 0 > 4x$

$\Leftrightarrow x < 0$

Hence, in this case, the solutions are the values of x satisfying $x < 0$.

Hence, overall the solutions are the values of x satisfying $x < 0$ or $x > \frac{3}{2}$. (5)

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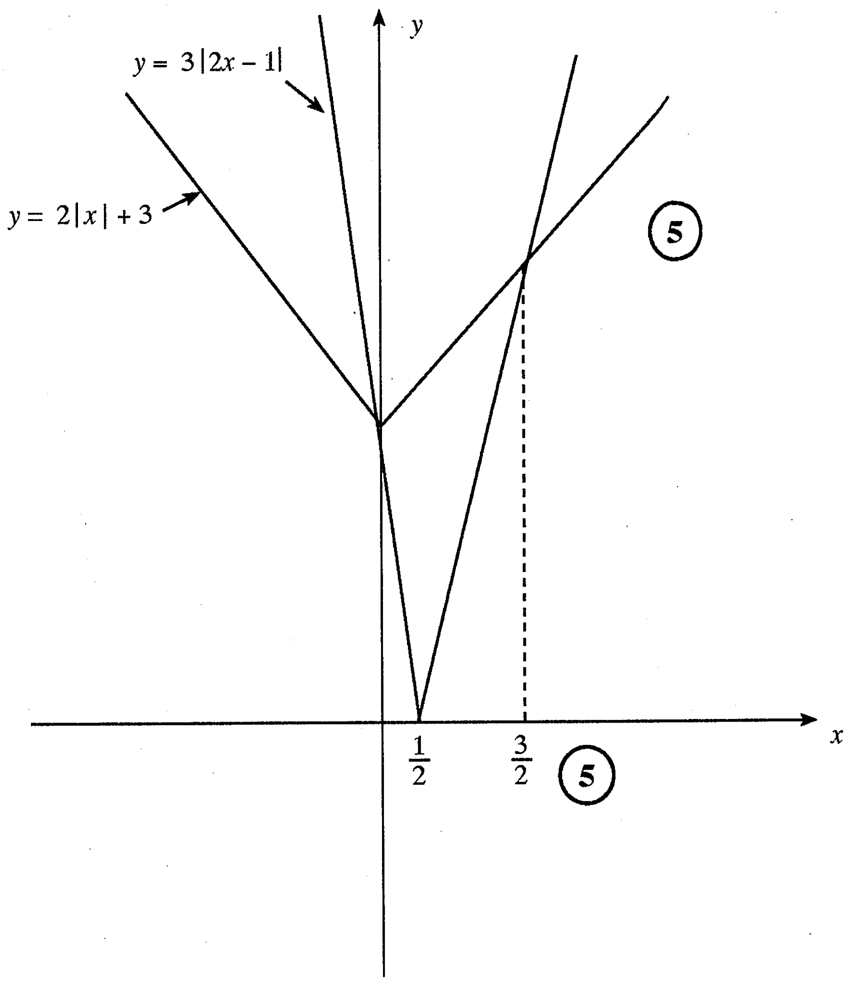
Aliter 2:



From the graphs,

$3|2x-1| > 2|x|+3$

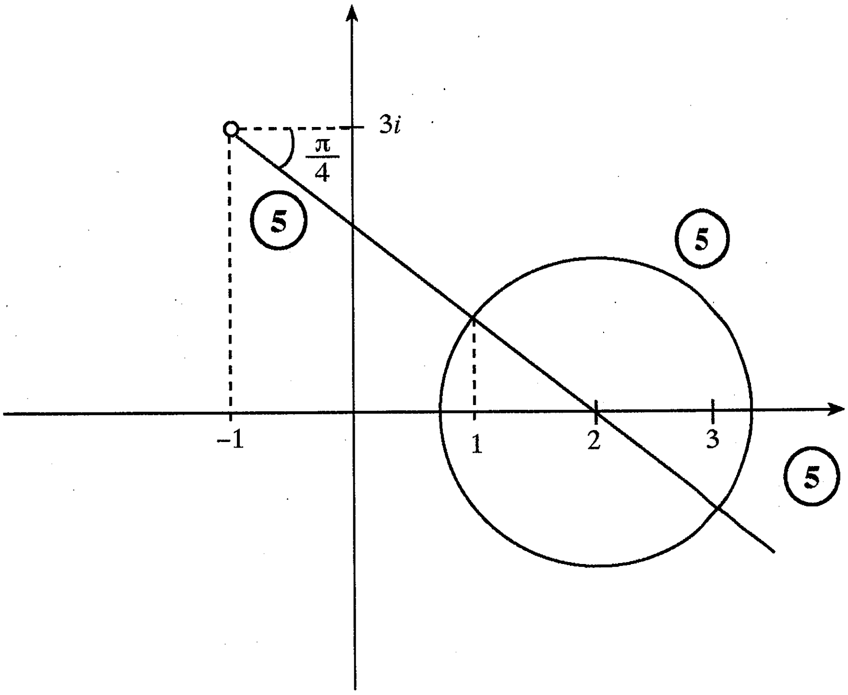
$\Leftrightarrow \quad x < 0$ or $x > \frac{3}{2}$. (5)

3. Sketch, in the same Argand diagram, the loci of the points that represent the complex numbers z satisfying

(i) $\text{Arg}(z+1-3i) = -\frac{\pi}{4}$ and

(ii) $|z-2| = \sqrt{2}$.

Hence, write down the complex numbers represented by the points of intersection of these loci.



The required complex numbers are $1 + i$ (5) and $3 - i$. (5)

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4. Let $n \in \mathbb{Z}^+$. Write down the binomial expansion of $(1+x)^n$ in ascending powers of x.
Show that if the coefficients of two consecutive terms of the above expansion are equal, then n is odd.

$$(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r, \text{ where } {}^nC_r = \frac{n!}{r!\,(n-r)!} \text{ for } r = 1, 2, \ldots, n,$$

(5) (5)

$$\text{and } {}^nC_0 = 1.$$

Two consecutive terms can be taken as

${}^nC_r$ and ${}^nC_{r+1}$

${}^nC_r = {}^nC_{r+1}$ (5) for some $r \in \{0, 1, \ldots n-1\}$

$\Leftrightarrow \dfrac{n!}{r!\,(n-r)!} = \dfrac{n!}{(r+1)!\,(n-r-1)!}$ (5)

$\Leftrightarrow \dfrac{1}{n-r} = \dfrac{1}{r+1}$

$\Leftrightarrow n - r = r + 1$

$\Leftrightarrow n = 2r + 1.$

$\therefore n$ is odd. (5)

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Aliter :

Two consecutive terms can be taken as ${}^nC_{r-1}$ and ${}^nC_r$

${}^nC_{r-1} = {}^nC_r$ (5) for some $r \in \{1, 2, 3 \ldots, n\}$

$\Leftrightarrow \dfrac{n!}{[n-(r-1)]!\,(r-1)!} = \dfrac{n!}{(n-r)!\,r!}$ (5)

$\Leftrightarrow \dfrac{1}{n-(r-1)} = \dfrac{1}{r}$

$\Leftrightarrow n - r + 1 = r$

$\Leftrightarrow n = 2r - 1.$

$\therefore n$ is odd. (5)

5. Show that $\lim\limits_{x\to\frac{\pi}{3}} \dfrac{\sin\left(x-\frac{\pi}{3}\right)}{\left(\sqrt{3x}-\sqrt{\pi}\right)} = \dfrac{2\sqrt{\pi}}{3}$.

$$\lim_{x\to\frac{\pi}{3}} \frac{\sin(x-\frac{\pi}{3})}{(\sqrt{3x}-\sqrt{\pi})}$$

$$= \lim_{x\to\frac{\pi}{3}} \frac{\sin(x-\frac{\pi}{3})}{(\sqrt{3x}-\sqrt{\pi})} \times \frac{(\sqrt{3x}+\sqrt{\pi})}{(\sqrt{3x}+\sqrt{\pi})} \quad (5)$$

$$= \lim_{x\to\frac{\pi}{3}} \frac{\sin(x-\frac{\pi}{3})}{(3x-\pi)} \cdot (\sqrt{3x}+\sqrt{\pi}) \quad (5)$$

$$= \lim_{x\to\frac{\pi}{3}} \frac{\sin(x-\frac{\pi}{3})}{3(x-\frac{\pi}{3})} \cdot \lim_{x\to\frac{\pi}{3}} (\sqrt{3x}+\sqrt{\pi})$$

$$= \frac{1}{3} \lim_{u\to 0} \frac{\sin u}{u} \cdot (\sqrt{\pi}+\sqrt{\pi}) \quad (5) \quad (5)$$

$$= \frac{1}{3} \cdot 1 \cdot 2\sqrt{\pi} = \frac{2\sqrt{\pi}}{3} \quad (5)$$

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Aliter:

$$\lim_{x\to\frac{\pi}{3}} \frac{\sin(x-\frac{\pi}{3})}{(\sqrt{3x}-\sqrt{\pi})}$$

$$= \lim_{x\to\frac{\pi}{3}} \frac{\sin(x-\frac{\pi}{3})}{(x-\frac{\pi}{3})} \times \frac{(x-\frac{\pi}{3})}{\sqrt{x}-\sqrt{\frac{\pi}{3}}} \times \frac{1}{\sqrt{3}} \quad (5)$$

$$= \left[\lim_{x\to\frac{\pi}{3}} \frac{\sin(x-\frac{\pi}{3})}{(x-\frac{\pi}{3})}\right] \cdot \left[\lim_{x\to\frac{\pi}{3}} \frac{\left(\sqrt{x}-\sqrt{\frac{\pi}{3}}\right)\left(\sqrt{x}+\sqrt{\frac{\pi}{3}}\right)}{\left(\sqrt{x}-\sqrt{\frac{\pi}{3}}\right)}\right] \frac{1}{\sqrt{3}}$$

(5) (5)

$$= 1 \cdot \frac{2\sqrt{\pi}}{3} \cdot \frac{1}{\sqrt{3}} \quad (5)$$

$$= \frac{2\sqrt{\pi}}{3} \quad (5)$$

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6. The region enclosed by the curves $y = \frac{e^x}{1+e^x}$, $x = 0$, $x = \ln 3$ and $y = 0$ is rotated about the x–axis through 2π radians. Show that the volume of the solid thus generated is $\frac{\pi}{4}(4\ln 2 - 1)$.

The required volume $= \pi \int_0^{\ln 3} \frac{e^{2x}}{(1+e^x)^2}\, dx$ (5)

$= \pi \int_2^4 \frac{u-1}{u^2}\, du$ Let $u = 1 + e^x$. (5)

$= \pi \int_2^4 \left\{\frac{1}{u} - \frac{1}{u^2}\right\} du$ (5)

$= \pi \left\{\ln|u| + \frac{1}{u}\right\}\Big|_2^4$ (5)

$= \pi \left\{\ln 4 - \ln 2 + \frac{1}{4} - \frac{1}{2}\right\}$

$= \frac{\pi}{4}\{4\ln 2 - 1\}$ (5)

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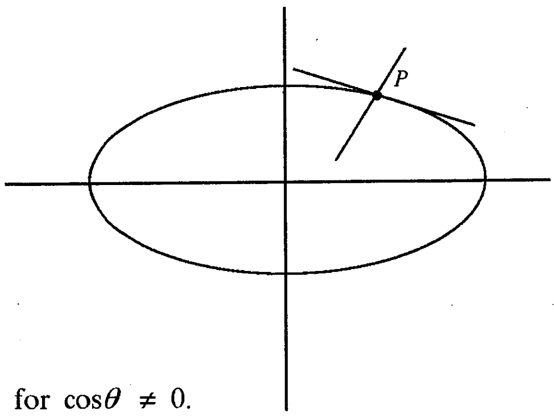
7. Show that the equation of the normal line to the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ at the point $P \equiv (5\cos\theta,\ 3\sin\theta)$ on it, is $5\sin\theta\, x - 3\cos\theta\, y = 16\sin\theta\cos\theta$.

Find the y-intercept of the normal line drawn to the above ellipse at the point $\left(\frac{5}{2}, \frac{3\sqrt{3}}{2}\right)$ on it.



$x = 5\cos\theta,\ y = 3\sin\theta$

$\frac{dx}{d\theta} = -5\sin\theta,\ \frac{dy}{d\theta} = 3\cos\theta$ (5)

$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3\cos\theta}{-5\sin\theta}$ (5) for $\sin\theta \neq 0$.

$\therefore$ The gradient of the normal at $P = \frac{5\sin\theta}{3\cos\theta}$ (5) for $\cos\theta \neq 0$.

The required equation is

$$y - 3\sin\theta = \frac{5\sin\theta}{3\cos\theta}(x - 5\cos\theta) \quad \text{for } \cos\theta \neq 0.$$

$$3y\cos\theta - 9\sin\theta\cos\theta = 5x\sin\theta - 25\sin\theta\cos\theta$$

$$5\sin\theta\, x - 3\cos\theta\, y = 16\sin\theta\cos\theta.$$ (5)

The equation is valid even when $\cos\theta = 0$ (P lies on the y – axis).

For y – intercept : $y = -\frac{16}{3}\sin\theta$.

But $3\sin\theta = \frac{3\sqrt{3}}{2} \Rightarrow \sin\theta = \frac{\sqrt{3}}{2}$

$\therefore\ y = -\frac{8}{\sqrt{3}}$ (5)

$\left(0, -\frac{8}{\sqrt{3}}\right)$

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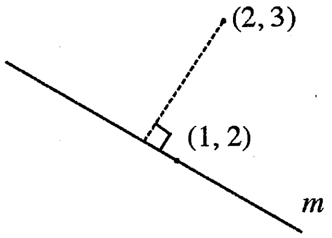
8. Let $m \in \mathbb{R}$ and l be the straight line passing through the point $A \equiv (1, 2)$ with gradient m. Write down the equation of l in terms of m.
It is given that the perpendicular distance from the point $B \equiv (2, 3)$ to the line l is $\frac{1}{\sqrt{5}}$ units. Find the values of m.



$$y - 2 = m(x - 1)$$

$$y - mx - 2 + m = 0$$ (5)

$$\frac{1}{\sqrt{5}} = \frac{|3 - 2m - 2 + m|}{\sqrt{1 + m^2}}$$ (5)

$\Leftrightarrow \quad 1 + m^2 = 5(1 - m)^2$ (5)

$\Leftrightarrow \quad 1 + m^2 = 5(1 - 2m + m^2)$

$\Leftrightarrow \quad 4m^2 - 10m + 4 = 0$

$\Leftrightarrow \quad 2m^2 - 5m + 2 = 0$

$\Leftrightarrow \quad (2m - 1)(m - 2) = 0$ (5)

$\Leftrightarrow \quad m = \frac{1}{2}$ or $m = 2.$

(5)

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9. Find the equation of the circle S having the centre at the point $(-2, 0)$ and passing through the point $(-1, \sqrt{3})$. Write down the equation of the chord of contact of the tangents drawn from the point $A \equiv (1, -1)$ to the circle S.
Hence, show that the x–coordinates of the points of contact of the tangents drawn to S from A satisfies the equation $5x^2 + 8x + 2 = 0$.

$S: (x+2)^2 + y^2 = r^2$ (5)

This goes through $(-1, \sqrt{3})$.

$\therefore \; 1 + 3 = r^2$.

$\therefore \; 4 = r^2$.

Hence, the equation of S is

$(x+2)^2 + y^2 = 4$. (5)

$\therefore \; x^2 + y^2 + 4x = 0$. ——— (1)

The chord of contact of the tangents drawn to S from $A \equiv (1, -1)$ is

$x - y + 2(x+1) = 0$.

i.e. $3x - y + 2 = 0$. (5)

For the points of contact, we substitute $y = 3x + 2$ in (1). (5)

i.e. $x^2 + (3x+2)^2 + 4x = 0$.

Hence, $10x^2 + 12x + 4 + 4x = 0$ and so

$5x^2 + 8x + 2 = 0$. (5)

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10. Let $\theta \neq (2n+1)\frac{\pi}{2}$ for $n \in \mathbb{Z}$.
Using the identity $\cos^2\theta + \sin^2\theta = 1$, show that $\sec^2\theta = 1 + \tan^2\theta$.
It is given that $\sec\theta + \tan\theta = \frac{4}{3}$. **Deduce** that $\sec\theta - \tan\theta = \frac{3}{4}$.
Hence, show that $\cos\theta = \frac{24}{25}$.

$\cos^2\theta + \sin^2\theta = 1$

$\theta \neq (2n+1)\frac{\pi}{2}$ gives us $\cos^2\theta \neq 0$

and hence, $1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$. (5)

$\therefore \sec^2\theta = 1 + \tan^2\theta$. (5)

Now, $\sec^2\theta - \tan^2\theta = 1$ gives us

$(\sec\theta - \tan\theta)(\sec\theta + \tan\theta) = 1$. (5)

Since $\sec\theta + \tan\theta = \frac{3}{4}$, (5)

$\sec\theta - \tan\theta = \frac{4}{3}$.

$\therefore 2\sec\theta = \frac{3}{4} + \frac{4}{3} = \frac{25}{12}$.

$\therefore \cos\theta = \frac{24}{25}$. (5)

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11. (*a*) Let $f(x)=x^2+px+c$ and $g(x)=2x^2+qx+c$, where $p, q \in \mathbb{R}$ and $c>0$. It is given that $f(x)=0$ and $g(x)=0$ have a common root α. Show that $\alpha=p-q$.

Find c in terms of p and q, and **deduce** that

(i) if $p>0$, then $p<q<2p$,

(ii) the discriminant of $f(x)=0$ is $(3p-2q)^2$.

Let β and γ be the other roots of $f(x)=0$ and $g(x)=0$ respectively. Show that $\beta=2\gamma$.

Also, show that the quadratic equation whose roots are β and γ is given by

$2x^2+3(2p-q)x+(2p-q)^2=0$.

(*b*) Let $h(x)=x^3+ax^2+bx+c$, where $a, b, c \in \mathbb{R}$. It is given that x^2-1 is a factor of $h(x)$. Show that $b=-1$.

It is also given that the remainder when $h(x)$ is divided by x^2-2x is $5x+k$, where $k \in \mathbb{R}$. Find the value of k and show that $h(x)$ can be written in the form $(x-\lambda)^2(x-\mu)$, where $\lambda, \mu \in \mathbb{R}$.

(*a*) Since α is a common root of $f(x)=0$ and $g(x)=0$, we have

$\alpha^2+p\alpha+c=0$ —— (1) and (5) $\alpha^2+q\alpha+c=0$. (5)

$\therefore\ \alpha^2+(q-p)\alpha=0$ (5) and so $\alpha[\alpha-(p-q)]=0$

Hence, $\alpha=p-q$. (5) $(\because\ c>0 \Rightarrow \alpha\neq 0)$

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(1) $\Rightarrow c = -\alpha(\alpha+p)$ (5)

$= -(p-q)(2p-q)$ (5) By substituting for α

$= -(q-p)(q-2p)$.

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(i) $c>0 \Rightarrow (q-p)(q-2p)<0$ (5)

$\Rightarrow q$ lies between p and $2p$.

Assume that $p>0$. Then $p<2p$.

$\therefore\ p<q<2p$. (5)

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(ii) $\Delta = p^2 - 4c.$ (5)

$= p^2 + 4(q-p)\ (q-2p)$ (5)

$= p^2 + 4[q^2 - 3pq + 2p^2]$

$= 9p^2 - 12pq + 4p^2$

$= (3p-2q)^2.$ (5)

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$\alpha + \beta = -p.$ (5)

$\alpha + \gamma = -\dfrac{q}{2}.$ (5)

$\therefore \beta - 2\gamma = -p - \alpha + q + 2\alpha$

$= -p + q + \alpha$

$= 0.$ (5) $(\because \alpha = p - q)$

$\therefore \beta = 2\gamma$

Aliter

$\alpha\beta = c$ (5)

$\alpha r = \dfrac{c}{2}$ (5)

Since $\alpha, \beta, \gamma \neq 0$

$\dfrac{\beta}{\gamma} = 2$ (5)

$\beta = 2\gamma$

15

The required equation is $(x-\beta)\ (x-\gamma) = 0.$

This gives us $x^2 - (\beta+\gamma)x + \gamma\beta = 0.$ (5)

Also, $\beta + \gamma = -p - \dfrac{q}{2} - 2\alpha = -p - \dfrac{q}{2} - (2p-2q) = \dfrac{3}{2}\ (q-2p).$ (5)

Now, $\alpha^2\beta\gamma = \dfrac{c^2}{2}.$

$\therefore \beta\gamma = \dfrac{c^2}{2(p-q)^2} = \dfrac{(q-p)^2(q-2p)^2}{2(p-q)^2} = \dfrac{1}{2}\ (q-2p)^2.$ (5)

$x^2 - \dfrac{3}{2}\ (q-2p)\,x + \dfrac{1}{2}\ (q-2p)^2 = 0.$ (5)

$2x^2 + 3(2p-q)\,x + (2p-q)^2 = 0.$ (5)

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(*b*) Since $(x^2 - 1)$ is a factor of $h(x)$,

$(x - 1)$ and $(x + 1)$ are both factors of $h(x)$.

Factor theorem gives, $h(1) = 0$ and $h(-1) = 0$. (5)

$h(x) = x^3 + ax^2 + bx + c$.

$\therefore h(1) = 1 + a + b + c = 0$ —— ① (5) and $h(-1) = -1 + a - b + c = 0$. —— ② (5)

By ① − ②, we get; $2 + 2b = 0$.

$\therefore b = -1$. (5)

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$h(x) = p(x) . (x^2 - 2x) + 5x + k$ (5)

$h(0) = k$. (5)

$h(2) = 8 + 4a + 2(-1) + c = 10 + k$ (5)

$\therefore k = c$.

$4a + c = 4 + k$

$a = 1$ (5)

By ① + ②, we get; $a = -c$.

$\therefore c = -1$.

Hence, $k = -1$. (5)

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$h(x) = x^3 + x^2 - x - 1$

$= (x + 1)x^2 - (x + 1)$

$= (x + 1)(x^2 - 1)$

$= (x + 1)^2 (x - 1)$. (5)

$\lambda = -1$, $\mu = 1$. (5)

10

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12. (*a*) It is required to select a musical group consisting of eleven members from among five pianists, five guitarists, three female singers and seven male singers such that it includes **exactly** two pianists and **at least** four guitarists. Find the number of different such musical groups that can be selected.

Find also the number of musical groups among these, having exactly two female singers.

(*b*) Let $U_r = \dfrac{3r-2}{r(r+1)(r+2)}$ and $V_r = \dfrac{A}{r+1} - \dfrac{B}{r}$ for $r \in \mathbb{Z}^+$, where $A, B \in \mathbb{R}$.

Find the values of A and B such that $U_r = V_r - V_{r+1}$ for $r \in \mathbb{Z}^+$.

Hence, show that $\sum_{r=1}^{n} U_r = \dfrac{n^2}{(n+1)(n+2)}$ for $n \in \mathbb{Z}^+$.

Show that the infinite series $\sum_{r=1}^{\infty} U_r$ is convergent and find its sum.

Now, let $W_r = U_{r+1} - 2U_r$ for $r \in \mathbb{Z}^+$. Show that $\sum_{r=1}^{n} W_r = U_{n+1} - U_1 - \sum_{r=1}^{n} U_r$.

Deduce that the infinite series $\sum_{r=1}^{\infty} W_r$ is convergent and find its sum.

12. (*a*) P = Pianists (5), G = Guitarists (5), Singers (10)

FS – Female Singers (3)

MS – Male Singers (7)

P	G	S	Number of ways
2	4	5	⑩ ${}^5C_2\ {}^5C_4\ {}^{10}C_5 = 12600$ ⑤
2	5	4	⑩ ${}^5C_2\ {}^5C_5\ {}^{10}C_4 = 2100$ ⑤

The required number of ways $= 12600 + 2100$

$= 14700$ ⑤

35

P	G	FS	MS	Number of ways
2	4	2	3	(10) ${}^5C_2\,{}^5C_4\,{}^3C_2\,{}^7C_3 = 5250$ (5)
2	5	2	2	(10) ${}^5C_2\,{}^5C_5\,{}^3C_2\,{}^7C_2 = 630$ (5)

The required number of ways $= 5250 + 630$

$= 5880$ (5)

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(*b*) For $r \in \mathbb{Z}^+$;

$$U_r = \frac{3r-2}{r(r+1)(r+2)} \quad \text{and} \quad V_r = \frac{A}{(r+1)} - \frac{B}{r}.$$

Thus, $U_r = V_r - V_{r+1}$ gives us $\dfrac{3r-2}{r(r+1)(r+2)} = \dfrac{A}{r+1} - \dfrac{B}{r} - \dfrac{A}{r+2} + \dfrac{B}{r+1}.$ (5)

$$\therefore \frac{3r-2}{r(r+1)(r+2)} = \frac{A}{(r+1)(r+2)} - \frac{B}{r(r+1)} \text{ and}$$

hence, $3r - 2 = Ar - B(r+2)$ for $r \in \mathbb{Z}^+$.

(5)

Comparing coefficients of powers of r :

r^1 : $3 = A - B$; $A = 4$ (5)

r^0 : $-2 = -2B$; $B = 1$ (5)

20

$$U_r = V_r - V_{r+1}$$

$r = 1;\quad U_1 = V_1 - V_2$

$r = 2;\quad U_2 = V_2 - V_3$ (5)

$\vdots \quad \vdots$

$r = n-1;\quad U_{n-1} = V_{n-1} - V_n$

$r = n;\quad U_n = V_n - V_{n+1}$ (5)

$$\sum_{r=1}^{n} U_r = V_1 - V_{n+1} \quad (5)$$

$$= 1 - \left(\frac{4}{(n+2)} - \frac{1}{(n+1)}\right) \quad (5)$$

$$= \frac{n^2}{(n+1)(n+2)}. \quad (5)$$

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$$\lim_{n\to\infty} \sum_{r=1}^{n} U_r = \lim_{n\to\infty} \left\{\frac{n^2}{(n+1)(n+2)}\right\} \quad (5)$$

$$= \lim_{n\to\infty} \left\{\frac{1}{(1+\frac{1}{n})\ (1+\frac{2}{n})}\right\}$$

$$= 1. \quad (5)$$

Therefore, the infinite series $\sum_{r=1}^{\infty} U_r$ is convergent and the sum is 1.

(5)

15

$$W_r = U_{r+1} - 2U_r$$

$$\sum_{r=1}^{n} W_r = \sum_{r=1}^{n} (U_{r+1} - 2U_r)$$

$$= \sum_{r=1}^{n} U_r - U_1 + U_{n+1} - 2\sum_{r=1}^{n} U_r \quad (5)$$

$$= U_{n+1} - U_1 - \sum_{r=1}^{n} U_r. \quad (5)$$

10

$$\lim_{n\to\infty} \sum_{r=1}^{n} W_r = \lim_{n\to\infty} U_{n+1} - U_1 - \lim_{n\to\infty} \sum_{r=1}^{n} U_r$$

$$= 0 - \frac{1}{6} - 1 \quad (5)$$

$$= -\frac{7}{6}.$$

$\therefore \sum_{r=1}^{\infty} W_r$ is convergent and the sum is $-\frac{7}{6}$. (5)

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13. (*a*) Let $\mathbf{A} = \begin{pmatrix} a+1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a & 2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} a & 1 \\ a & 2 \end{pmatrix}$, where $a \in \mathbb{R}$.

Show that $\mathbf{A}^T\mathbf{B} - \mathbf{I} = \mathbf{C}$; where $\mathbf{I}$ is the identity matrix of order 2.

Show also that $\mathbf{C}^{-1}$ exists **if and only if** $a \neq 0$.

Now, let $a = 1$. Write down $\mathbf{C}^{-1}$.

Find the matrix $\mathbf{P}$ such that $\mathbf{CPC} = 2\mathbf{I} + \mathbf{C}$.

(*b*) Let $z, w \in \mathbb{C}$. Show that $|z|^2 = z\bar{z}$ and applying it to $z - w$,

show that $|z-w|^2 = |z|^2 - 2\,\text{Re}\, z\bar{w} + |w|^2$.

Write a similar expression for $|1 - z\bar{w}|^2$ and show that $|z-w|^2 - |1-z\bar{w}|^2 = -\left(1-|z|^2\right)\left(1-|w|^2\right)$.

Deduce that if $|w| = 1$ and $z \neq w$, then $\left|\dfrac{z-w}{1-z\bar{w}}\right| = 1$.

(*c*) Express $1 + \sqrt{3}i$ in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $0 < \theta < \frac{\pi}{2}$.

It is given that $(1+\sqrt{3}i)^m(1-\sqrt{3}i)^n = 2^8$, where m and n are positive integers.

Applying De Moivre's theorem, obtain equations sufficient to determine the values of m and n.

(*a*) $A^TB = \begin{bmatrix} a+1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{2\times 3} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ a & 2 \end{bmatrix}_{3\times 2}$ (5)

$= \begin{bmatrix} a+1 & 1 \\ a & 3 \end{bmatrix}$ (5)

$\therefore A^TB - I = \begin{bmatrix} a+1 & 1 \\ a & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (5)

$= \begin{bmatrix} a & 1 \\ a & 2 \end{bmatrix} = C$ (5)

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C^{-1} exists

$\Leftrightarrow |C| \neq 0$ (5)

$\Leftrightarrow 2a - a \neq 0$

$\Leftrightarrow a \neq 0$ (5)

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When $a = 1$, $C = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. (5)

$\therefore\ C^{-1} = \dfrac{1}{2-1}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$. (5)

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$CPC = 2I + C$

$\Leftrightarrow PC = 2C^{-1} + C^{-1}C$ (5)

$\Leftrightarrow PC = 2C^{-1} + I$

$\Leftrightarrow P = 2C^{-1}C^{-1} + C^{-1}$ (5)

$\therefore\ P = 2\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$= 2\begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ (5)

$= \begin{bmatrix} 10 & -6 \\ -6 & 4 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 12 & -7 \\ -7 & 5 \end{bmatrix}$ (5)

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(b) Let $z = x + iy$.

$z\overline{z} = (x + iy)(x - iy)$ (5)

$= x^2 + i^2y^2)$

$= x^2 + y^2$

$= |z|^2$

$\therefore\ |z|^2 = z\overline{z}$. (5)

10

$|z-w|^2$

$= (z-w)\,\overline{(z-w)}$ (5)

$= (z-w)\,(\overline{z}-\overline{w})$ (5)

$= z\overline{z} - z\overline{w} - \overline{z}w + w\overline{w}$

$= |z|^2 - (z\overline{w} + \overline{z\overline{w}}) + |w|^2$ (5)

$= |z|^2 - 2\,\text{Re}\,(z\overline{w}) + |w|^2 \longrightarrow ①$

15

$|1-z\overline{w}|^2$

$= 1 - 2\,\text{Re}\,(z\overline{w}) + |z\overline{w}|^2 \longrightarrow ②$ (5)

05

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① − ② gives;

$|z-w|^2 - |1-z\overline{w}|^2$

$= |z|^2 + |w|^2 - 1 - |z\overline{w}|^2$ (5)

$= -(1 - |w|^2 - |z|^2 + |z|^2|w|^2)$ (5)

$= -(1-|z|^2)(1-|w|^2)$ (5)

15

$|w| = 1,\ z \neq w$

$\Rightarrow |z-w|^2 - |1-z\overline{w}|^2 = 0$ (5)

$\Rightarrow |z-w| = |1-z\overline{w}|$

$\Rightarrow \dfrac{|z-w|}{|1-z\overline{w}|} = 1$ $\left[\because z \neq w \Rightarrow z\overline{w} \neq 1\right]$

$\Rightarrow \left|\dfrac{z-w}{1-z\overline{w}}\right| = 1$ (5)

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(*c*) $1 + \sqrt{3}\, i \; = 2\left\{ \frac{1}{2} + i\frac{\sqrt{3}}{2} \right\}$ (5)

$= 2\left\{ \cos\frac{\pi}{3} + i\,\sin\frac{\pi}{3} \right\}$ (5)

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$(1+\sqrt{3}\, i)^m\,(1-\sqrt{3}\, i)^n = 2^m\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^m 2^n\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)^n$ (5)

$= 2^{m+n}\left(\cos\frac{m\pi}{3} + i\sin\frac{m\pi}{3}\right)\left(\cos\left(-\frac{n\pi}{3}\right) + i\sin\left(-\frac{n\pi}{3}\right)\right)$ (5)

$= 2^{m+n}\left(\cos(m-n)\frac{\pi}{3} + i\sin(m-n)\frac{\pi}{3}\right)$ (5)

$\therefore\; 2^{m+n}\left(\cos(m-n)\frac{\pi}{3} + i\sin(m-n)\frac{\pi}{3}\right) = 2^8$

$\Rightarrow m+n=8$ and $(m-n)\frac{\pi}{3} = 2k\pi$; $k \in \mathbb{Z}$.

(5) (5)

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14. (*a*) Let $f(x)=\dfrac{x(2x-3)}{(x-3)^2}$ for $x \neq 3$.

Show that $f'(x)$, the derivative of $f(x)$, is given by $f'(x)=\dfrac{9(1-x)}{(x-3)^3}$ for $x \neq 3$.

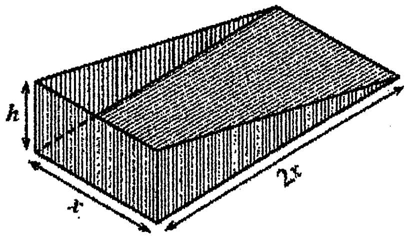
Hence, find the interval on which $f(x)$ is increasing and the intervals on which $f(x)$ is decreasing. Also, find the coordinates of the turning point of $f(x)$.

It is **given that** $f''(x)=\dfrac{18x}{(x-3)^4}$ for $x \neq 3$.

Find the coordinates of the point of inflection of the graph of $y=f(x)$.

Sketch the graph of $y=f(x)$ indicating the asymptotes, the turning point and the point of inflection.

(*b*) The adjoining figure shows the portion of a dust pan without its handle. Its dimensions in centimetres, are shown in the figure. It is given that its volume x^2h cm³ is 4500 cm³. Its surface area S cm² is given by $S = 2x^2 + 3xh$. Show that S is minimum when $x = 15$.



(*a*) For $x \neq 3$; $f(x) = \dfrac{x(2x-3)}{(x-3)^2}$

Then, $f'(x) = \dfrac{1}{(x-3)^2}\left[2x-3+2x\right] - \dfrac{2x(2x-3)}{(x-3)^3}$ **(20)**

$= \dfrac{(x-3)(4x-3)-2x(2x-3)}{(x-3)^3}$

$= \dfrac{4x^2-15x+9-4x^2+6x}{(x-3)^3}$

$= \dfrac{9(1-x)}{(x-3)^3}$. **(5)**

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$f'(x) = 0 \Leftrightarrow x = 1$. **(5)**

	$-\infty < x < 1$	$1 < x < 3$	$3 < x < \infty$
Sign of $f'(x)$	(–)	(+)	(–)
$f(x)$ is	↘ Decreasing	↗ Increasing	↘ Decreasing

(5) **(5)** **(5)**

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Turning point : $\left(1, -\frac{1}{4}\right)$ is a local minimum. (5)

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For $x \neq 3$; $f''(x) = \frac{18x}{(x-3)^4}$.

$f''(x) = 0 \Leftrightarrow x = 0.$ (5)

	$-\infty < x < 0$	$0 < x < 3$	$3 < x < \infty$
Sign of $f''(x)$	(−)	(+)	(+)
Concavity	Concave down	Concave up	Concave up

(10)

$\therefore$ Point of inflection = (0, 0). (5)

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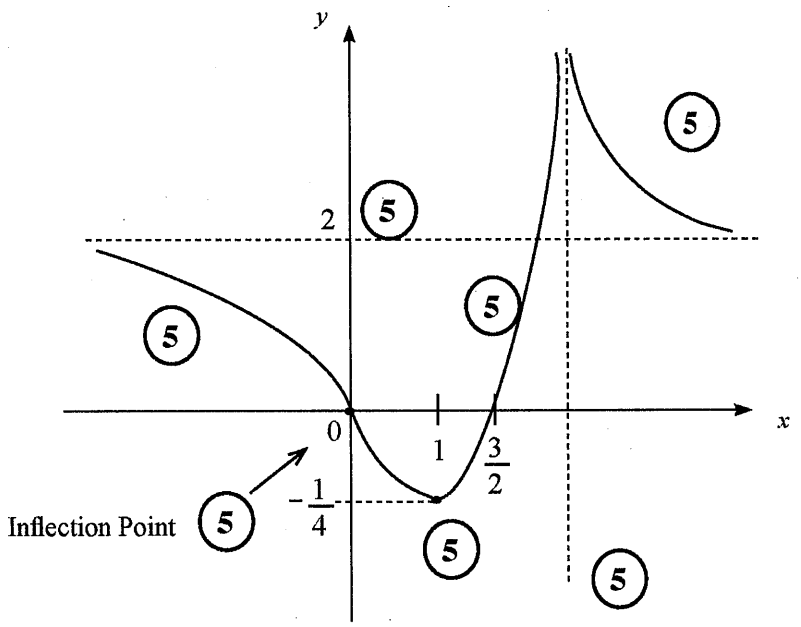
Horizontal asymptote : $\lim_{x \to \pm\infty} f(x) = 2 \quad \therefore y = 2$ (5)

Vertical asymptote : $x = 3.$ (5)



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(*b*) $x^2 h = 4500.$

Hence, $S = 2x^2 + 3xh$

$$= 2x^2 + 3x \cdot \frac{4500}{x^2} \quad \text{for } x > 0.$$ (5)

$$\therefore \frac{dS}{dx} = 4x - 3 \times 4500 \left(\frac{1}{x^2}\right) = \frac{4(x^3 - 3375)}{x^2}.$$ (5)

$\frac{dS}{dx} = 0$ (5) $\Leftrightarrow x = 15.$ (5)

For $0 < x < 15$, $\frac{ds}{dr} < 0$ (5) and for $x > 15$, $\frac{ds}{dr} > 0$. (5)

$\therefore S$ is minimum when $x = 15.$ (5)

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15. (*a*) It is given that there exist constants A and B such that

$x^3 + 13x - 16 = A(x^2+9)(x+1) + B(x^2+9) + 2(x+1)^2$ for all $x \in \mathbb{R}$.

Find the values of A and B.

Hence, write down $\dfrac{x^3+13x-16}{(x+1)^2(x^2+9)}$ in partial fractions and

find $\displaystyle\int \frac{x^3+13x-16}{(x+1)^2(x^2+9)}\,dx$.

(*b*) Using integration by parts, evaluate $\displaystyle\int_0^1 e^x \sin^2 \pi x\, dx$.

(*c*) Using the formula $\displaystyle\int_0^a f(x)dx = \int_0^a f(a-x)dx$, where a is a constant,

show that $\displaystyle\int_0^\pi x\cos^6 x \sin^3 x\, dx = \frac{\pi}{2}\int_0^\pi \cos^6 x \sin^3 x\, dx$.

Hence, show that $\displaystyle\int_0^\pi x\cos^6 x \sin^3 x\, dx = \frac{2\pi}{63}$.

(*a*) All $x \in \mathbb{R}$

$x^2 + 13x - 16 = A(x^2+9)(x+1) + B(x^2+9) + 2(x+1)^2$

Comparing coefficients of powers of x;

x^3 : $1 = A$. (5)

x^0 : $-16 = 9A + 9B + 2 \Rightarrow B = -3$.
(5) (5)

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$$\therefore \frac{x^2+13x-16}{(x+1)^2(x^2+9)} = \frac{1}{(x+1)} - \frac{3}{(x+1)^2} + \frac{2}{x^2+9}. \quad (10)$$

$$\int \frac{x^2+13x-16}{(x+1)^2(x^2+9)}\,dx = \int \frac{1}{x+1}\,dx - 3\int \frac{1}{(x+1)^2}\,dx + 2\int \frac{1}{x^2+9}\,dx$$

$$= \ln|x+1| + \frac{3}{x+1} + \frac{2}{3}\tan^{-1}\left(\frac{x}{3}\right) + C.$$

(5) (5) (5) (5)

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(b) $\displaystyle\int_0^1 e^x \sin^2 \pi x \ \ dx = \frac{1}{2}\int_0^1 e^x(1 - \cos^2\pi x) \ dx$ (5)

$$= \frac{1}{2} e^x \Big|_0^1 - \frac{1}{2}\underbrace{\int_0^1 e^x \cos 2\pi x \ dx}_{I} \quad (5)$$

$$= \frac{1}{2}(e-1) - \frac{1}{2} I. \quad (5) \qquad \text{———— ①}$$

Now, $\displaystyle I = \int_0^1 e^x \cos 2\pi x \quad dx$

$$= \underbrace{e^x \frac{\sin 2\pi x}{2\pi}\Big|_0^1}_{(5)} - \frac{1}{2\pi}\int_0^1 e^x \sin 2\pi x \ \ dx \quad (5)$$

$$= 0 \ (5) - \frac{1}{2\pi}\left[\left(-e^x \frac{\cos 2\pi x}{2\pi}\right)\Big|_0^1 \ (5) + \frac{1}{2\pi}\underbrace{\int_0^1 e^x \cos 2\pi x \ \ dx}_{I}\right]$$

$$= \frac{1}{4\pi^2}\left[e - 1\right] \ (5) - \frac{1}{4\pi^2} I. \quad (5)$$

$$\therefore I \left(1 + \frac{1}{4\pi^2}\right) = \frac{1}{4\pi^2}(e-1).$$

$$\therefore I = \frac{(e-1)}{4\pi^2 + 1}. \quad (5)$$

$$\therefore \text{ By ①, } \int_0^1 e^x \sin^2 \pi x \ \ dx \ \ \frac{1}{2}(e-1) - \frac{1}{2}\frac{(e-1)}{(4\pi^2+1)} \quad (5)$$

$$= \frac{(e-1)}{2}\left[\frac{4\pi^2}{4\pi^2+1}\right]$$

$$= \frac{2(e-1)\pi^2}{1 + 4\pi^2}. \quad (5)$$

60

(c) $I = \int_0^{\pi} x\cos^6 x \sin^3 x \, dx$

$= \int_0^{\pi} (\pi - x) \underbrace{\cos^6(\pi - x)}_{\cos^6 x} \underbrace{\sin^3(\pi - x)}_{\sin^3 x} dx = \int_0^{\pi} (\pi - x)\cos^6 x \sin^3 x$ (5) (5)

$= \pi\int_0^{\pi} \cos^6 x \sin^3 x \, dx - \underbrace{\int_0^{\pi} x\cos^6 x \sin^3 x \, dx}_{I}$ (5)

$\therefore I = \frac{\pi}{2}\int_0^{\pi} \cos^6 x \sin^3 x \, dx$. (5)

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$I = \frac{\pi}{2}\int_0^{\pi} \cos^6 x \sin^3 x \, dx$

$= \frac{\pi}{2}\int_0^{\pi} \cos^6 x \sin^2 x \sin x \, dx$

$= \frac{\pi}{2}\int_0^{\pi} \cos^6 x (1 - \cos^2 x)\sin x \, dx$ (5)

$= \frac{\pi}{2}\left[\int_0^{\pi} \cos^6 x \sin x \, dx - \int_0^{\pi} \cos^8 x \sin x \, dx\right]$ (5)

$= \frac{\pi}{2}\left[\frac{-\cos^7 x}{7}\Big|_0^{\pi} + \frac{\cos^9 x}{9}\Big|_0^{\pi}\right]$ (5) (5)

$= \frac{\pi}{2}\left[\frac{2}{7} - \frac{2}{9}\right]$ (5)

$= \frac{2\pi}{63}$.

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16. Let $A \equiv (1, 2)$ and $B \equiv (3, 3)$.

Find the equation of the straight line l passing through the points A and B.

Find the equations of the straight lines l_1 and l_2 passing through A, each making an acute angle $\frac{\pi}{4}$ with l.

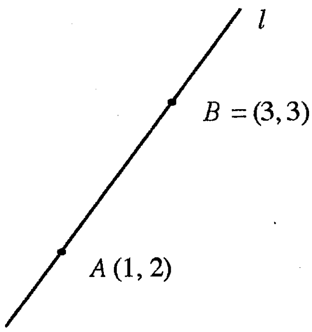
Show that the coordinates of any point on l can be written in the form $(1+2t, 2+t)$, where $t \in \mathbb{R}$.

Show also that the equation of the circle C_1 lying entirely in the first quadrant with radius $\frac{\sqrt{10}}{2}$, touching both l_1 and l_2, and its centre on l is $x^2+y^2-6x-6y+\frac{31}{2}=0$.

Write down the equation of the circle C_2 whose ends of a diameter are A and B.

Determine whether the circles C_1 and C_2 intersect orthogonally.

(16)



gradient $= \frac{3-2}{3-1} = \frac{1}{2}$ (5)

Equation of l: $y - 2 = \frac{1}{2}(x-1)$ (5)

i.e. $2y - 4 = x - 1$

i.e. $x - 2y + 3 = 0$ (10)

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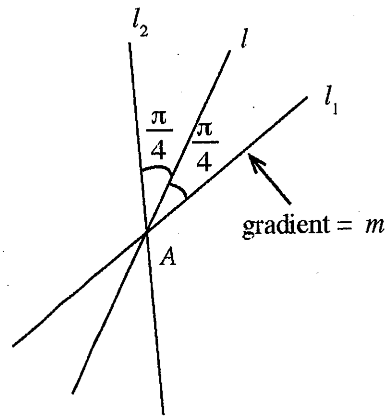
$$\tan\frac{\pi}{4} = \left|\frac{m - \frac{1}{2}}{1 + \frac{1}{2}m}\right| \quad (10)$$

$$\therefore \quad 1 = \left|\frac{2m-1}{2+m}\right| \quad (5)$$



$\Leftrightarrow \quad 2 + m = \pm(2m - 1)$

$\Leftrightarrow \quad 2 + m = 2m - 1$ or $2 + m = -2m + 1$

$\Leftrightarrow \quad m = 3$ or $m = -\frac{1}{3}$.

(5) (5)

$l_1 : y - 2 = 3(x - 1)$ and $l_2 : y - 2 = -\frac{1}{3}(x - 1)$.

$l_1 : 3x - y - 1 = 0$ and $l_2 : x + 3y - 7 = 0$.

(5) (5)

or vice versa. **35**

$l : \frac{x-1}{2} = \frac{y-2}{1} = t$ (say). (5)

Then $x = 1 + 2t,\ y = 2 + t$, where $t \in \mathbb{R}$. (5) **10**

For C_1,

The perpendicular distance to l_1 from $P \equiv (1 + 2t, 2 + t)$ is equal to the radius of C_1

i.e. $\frac{|3(1+2t) - (2+t) - 1|}{\sqrt{3^2 + (-1)^2}} = \frac{\sqrt{10}}{2}$. (10) (5)

i.e. $|3 + 6t - 2 - t - 1| = 5$. (5)

$|5t| = 5$.

$t = \pm 1$ (5)

$P \equiv (3, 3) = B$, since $P \equiv (-1, 1)$ is not suitable.

(5) (5)

$C_1 : (x - 3)^2 + (y - 3)^2 = \frac{5}{2}$. (5)

i.e. $x^2 + y^2 - 6x - 6y + 18 = \frac{5}{2}$

i.e. $x^2 + y^2 - 6x - 6y + \frac{31}{2} = 0$ (5) **45**

The equation of C_2 is

$(x - 1)(x - 3) + (y - 2)(y - 3) = 0$. (10)

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$$2g_1g_2 + 2f_1f_2 = 2(-3)(-2) + 2(-3)\left(-\frac{5}{2}\right) = 27.$$ (5)

(10)

$$c_1 + c_2 = \frac{31}{2} + 9 = \frac{49}{2}.$$ (5)

$\therefore\ 2g_1g_2 + 2f_1f_2 \neq c_1 + c_2.$ (5)

$\therefore$ C_1 and C_2 do not intersect orthogonally. (5)

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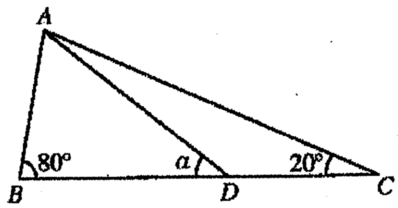
17. (*a*) Write down $\sin(A-B)$ in terms of $\sin A$, $\cos A$, $\sin B$ and $\cos B$.

Deduce that

(i) $\sin(90^\circ-\theta)=\cos\theta$, and

(ii) $2\sin 10^\circ=\cos 20^\circ-\sqrt{3}\sin 20^\circ$.

(*b*) In the usual notation, state the **Sine Rule** for a triangle ABC.



In the triangle ABC shown in the figure, $A\hat{B}C=80^\circ$ and $A\hat{C}B=20^\circ$. The point D lies on BC such that $AB=DC$. Let $A\hat{D}B=\alpha$.

Using the **Sine Rule** for suitable triangles, show that $\sin 80^\circ\sin(\alpha-20^\circ)=\sin 20^\circ\sin\alpha$.

Explain why $\sin 80^\circ=\cos 10^\circ$ and **hence**, show that $\tan\alpha=\dfrac{\sin 20^\circ}{\cos 20^\circ-2\sin 10^\circ}$.

Using the result in (*a*)(ii) above, **deduce** that $\alpha=30^\circ$.

(*c*) Solve the equation $\tan^{-1}(\cos^2 x)+\tan^{-1}(\sin x)=\frac{\pi}{4}$.

(*a*) $\sin(A-B) = \sin A\cos B-\cos A\sin B.$ (10) **[10]**

(i) $\sin(90^\circ-\theta) = \sin 90^\circ\cos\theta - \cos 90^\circ\sin\theta$ (5)

$= \cos\theta.$ (5) $(\because \sin 90^\circ = 1 \text{ and } \cos 90^\circ = 0.)$ **[10]**

(ii) $2\sin 10^\circ = 2\sin(30^\circ-20^\circ)$ (5)

$= 2\sin 30^\circ\cos 20^\circ - 2\cos 30^\circ\sin 20^\circ$ (5)

$= \cos 20^\circ - \sqrt{3}\sin 20^\circ.$ (5) $\left(\because \sin 30^\circ=\frac{1}{2} \text{ and } \cos 30^\circ=\frac{\sqrt{3}}{2}\right)$

[15]

(b) $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\quad}$, (10)

where $BC = a, CA = b$ and $AB = c$. **10**

Using the sine Rule :

for the trangle ABD; $\dfrac{AB}{\sin\alpha} = \dfrac{AD}{\sin 80°}$. (10)

for the trangle ADC; $\dfrac{DC}{\sin(\alpha - 20°)} = \dfrac{AD}{\sin 20°}$. (5)

$\therefore \dfrac{\sin(\alpha - 20°)}{\sin\alpha} = \dfrac{\sin 20°}{\sin 80°}$. (5)

$\therefore \sin 80° \sin(\alpha - 20°) = \sin 20° \sin\alpha$. (5)

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$\sin 80° = \sin(90° - 10°) = \cos 10°$ (5) (5)

Now, $\sin 80° \sin(\alpha - 20°) = \sin 20° \sin\alpha$ gives

$\cos 10° \sin(\alpha - 20°) = 2\sin 10° \cos 10° \sin\alpha$. (5) (5)

$\therefore \sin\alpha \cos 20° - \cos\alpha \sin 20° = 2\sin 10° \sin\alpha$. (5)

$\therefore \tan\alpha(\cos 20° - 2\sin 10°) = \sin 20°$ (5) and hence $\tan\alpha = \dfrac{\sin 20°}{\cos 20° - 2\sin 10°}$. (5)

35

By (a)(ii), we get $\tan\alpha = \dfrac{\sin 20°}{\sqrt{3}\,\sin 20°} = \dfrac{1}{\sqrt{3}}$. (5)

$\therefore\ \alpha = 30°$. (5) $(20° < \alpha < 90°)$

10

(c) $\tan^{-1}(\cos^2 x) + \tan^{-1}(\sin x) = \dfrac{\pi}{4}$.

Let $\alpha = \tan^{-1}(\cos^2 x)$ and $\beta = \tan^{-1}(\sin x)$

Then, $\alpha = \dfrac{\pi}{4} - \beta$.

$\therefore \tan\alpha = \tan\left(\dfrac{\pi}{4} - \beta\right)$ (5)

$= \dfrac{1 - \tan\beta}{1 + \tan\frac{\pi}{4}\ \tan\beta}$. (5)

$\Rightarrow \cos^2 x = \dfrac{1 - \sin x}{1 + \sin x}$. (5)

$\cos^2 x\,(1 + \sin x) = (1 - \sin x)$

$(1 - \sin^2 x)(1 + \sin x) = (1 - \sin x)$ (5)

$(1 - \sin x)(1 + \sin x)^2 = 1 - \sin x$

$\Rightarrow \sin x = 1$ or $1 + \sin x = \pm 1$

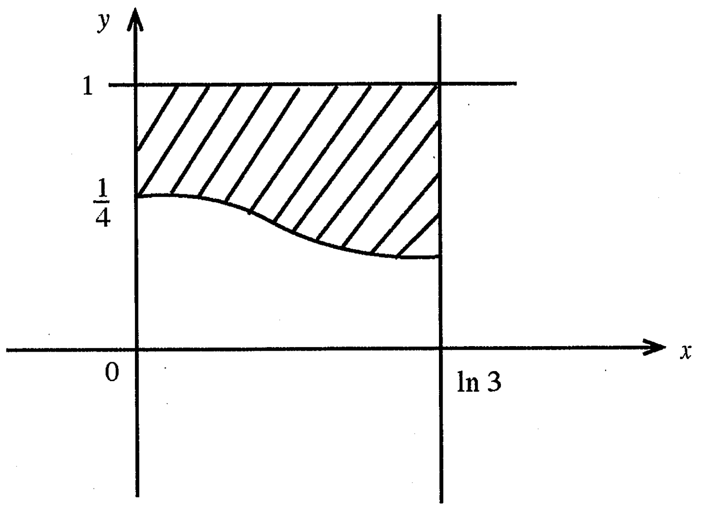
$\Rightarrow \sin x = 1$ or $\sin x = 0$ (5) $(\because \sin x \neq -2)$

$\Rightarrow x = n\pi + (-1)^n \dfrac{\pi}{2}$ for $n \in \mathbb{Z}$ (5) or $x = m\pi$ for $m \in \mathbb{Z}$ (5)

35

Old Syllabus

6. Show that the area of the region bounded by the curves $y = \dfrac{e^{2x}}{(1+e^x)^2}$, $x = 0, x = \ln 3$ and $y = 1$ is $\ln\left(\dfrac{3}{2}\right) + \dfrac{1}{4}$.



(5)

$$\text{Area} = \pi \int_0^{\ln 3} \left\{ 1 - \frac{e^{2x}}{(1+e^x)^2} \right\} dx$$

$$= \ln 3 - \int_2^4 \frac{u-1}{u^2}\, du \qquad (u = 1 + e^x.)$$

(5)

$$= \ln 3 - \int_2^4 \left\{ \frac{1}{u} - \frac{1}{u^2} \right\} du \quad (5)$$

$$= \ln 3 - \left\{ \ln|u| + \frac{1}{u} \right\} \Big|_2^4$$

(5)

$$= \ln 3 - \left\{ \ln 4 - \ln 2 + \frac{1}{4} - \frac{1}{2} \right\}$$

$$= \ln 3 - \left\{ \ln 2 - \frac{1}{4} \right\}$$

$$= \ln\left(\frac{3}{2}\right) + \frac{1}{4}. \quad (5)$$

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7. A curve C is given parametrically by $x = 2t - \cos 2t$ and $y = 1 - \sin 2t$ for $-\frac{\pi}{4} < t < \frac{3\pi}{4}$. Find $\frac{dy}{dx}$ in terms of t.
Show that the equation of the normal line drawn to the curve C at the point on it corresponding to $t = \frac{\pi}{12}$ is $6\sqrt{3}\,x - 6y - \sqrt{3}\,\pi + 12 = 0$.

$x = 2t - \cos 2t, \quad y = 1 - \sin 2t$

$\frac{dx}{dt} = 2 + 2\sin 2t, \quad \frac{dy}{dt} = -2\cos 2t.$ (5)

$\frac{dy}{dt} = \frac{-2\cos 2t}{2 + 2\sin 2t} = -\frac{\cos 2t}{1 + \sin 2t}$ (5)

$t = \frac{\pi}{12}$ gives us $x = \frac{\pi}{6} - \frac{\sqrt{3}}{2}$ and $y = 1 - \frac{1}{2} = \frac{1}{2}$. (5)

The gradient of the required normal $= \dfrac{1 + \sin\frac{\pi}{6}}{\cos\frac{\pi}{6}}$

$= \dfrac{\frac{3}{2}}{\frac{\sqrt{3}}{2}} = \sqrt{3}$ (5)

The required equation :

$$y - \frac{1}{2} = \sqrt{3}\left(x - \frac{\pi}{6} + \frac{\sqrt{3}}{2}\right)$$

i.e. $6y - 3 = 6\sqrt{3}\,x - \sqrt{3}\,\pi + 9$

i.e. $6\sqrt{3}\,x - 6y - \sqrt{3}\,\pi + 12 = 0.$ (5)

13. (*a*) Let $\mathbf{A} = \begin{pmatrix} a+1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a & 2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} a & 1 \\ a & 2 \end{pmatrix}$, where $a \in \mathbb{R}$.

Show that $\mathbf{A}^T\mathbf{B} - \mathbf{I} = \mathbf{C}$; where $\mathbf{I}$ is the identity matrix of order 2.

Show also that $\mathbf{C}^{-1}$ exists **if and only if** $a \neq 0$.

Now, let $a = 1$. Write down $\mathbf{C}^{-1}$.

Find the matrix $\mathbf{P}$ such that $\mathbf{CPC} = 2\mathbf{I} + \mathbf{C}$.

(*b*) Let $z, w \in \mathbb{C}$. Show that $|z|^2 = z\bar{z}$ and applying it to $z - w$,

show that $|z-w|^2 = |z|^2 - 2\,\mathrm{Re}\, z\bar{w} + |w|^2$.

Write a similar expression for $|1 - z\bar{w}|^2$ and show that $|z-w|^2 - |1-z\bar{w}|^2 = -\left(1-|z|^2\right)\left(1-|w|^2\right)$.

Deduce that if $|w| = 1$ and $z \neq w$, then $\left|\dfrac{z-w}{1-z\bar{w}}\right| = 1$.

(*c*) Express $1+\sqrt{3}i$ in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $0 < \theta < \frac{\pi}{2}$.
In an Argand diagram, point O represents the origin and point A represents the complex number $1+\sqrt{3}i$. Let $OABCDE$ be the regular hexagon having O and A as two of its consecutive vertices and the order of vertices taken in the counter clockwise sense. Find the complex numbers represented by the points B, C, D and E.

(*a*) $A^TB = \begin{bmatrix} a+1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{2\times 3} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ a & 2 \end{bmatrix}_{3\times 2}$ (5)

$= \begin{bmatrix} a+1 & 1 \\ a & 3 \end{bmatrix}$ (5)

$\therefore\ A^TB - I = \begin{bmatrix} a+1 & 1 \\ a & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (5)

$= \begin{bmatrix} a & 1 \\ a & 2 \end{bmatrix} = C$ (5)

20

C^{-1} exists

$\Leftrightarrow |C| \neq 0$ (5)

$\Leftrightarrow 2a - a \neq 0$

$\Leftrightarrow a \neq 0$ (5)

10

When $a = 1$, $C = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. (5)

$\therefore\ C^{-1} = \frac{1}{2-1}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$. (5)

10

$CPC = 2I + C$

$\Leftrightarrow PC = 2C^{-1} + C^{-1}C$ (5)

$\Leftrightarrow PC = 2C^{-1} + I$

$\Leftrightarrow P = 2C^{-1}C^{-1} + C^{-1}$ (5)

$\therefore\ P = 2\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$= 2\begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ (5)

$= \begin{bmatrix} 10 & -6 \\ -6 & 4 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 12 & -7 \\ -7 & 5 \end{bmatrix}$ (5)

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(*b*) Let $z = x + iy$.

$z\bar{z} = (x + iy)(x - iy)$ (5)

$= x^2 + i^2y^2)$

$= x^2 + y^2$

$= |z|^2$

$\therefore\ |z|^2 = z\bar{z}$. (5)

10

$|z-w|^2$

$= (z-w)\,\overline{(z-w)}$ (5)

$= (z-w)\,(\bar{z}-\bar{w})$ (5)

$= z\bar{z} - z\bar{w} - \bar{z}w + w\bar{w}$

$= |z|^2 - (z\bar{w} + \overline{z\bar{w}}) + |w|^2$ (5)

$= |z|^2 - 2\,\text{Re}\,(z\bar{w}) + |w|^2 \longrightarrow$ (1)

15

$|1 - z\bar{w}|^2$

$= 1 - 2\,\text{Re}\,(z\bar{w}) + |z\bar{w}|^2 \longrightarrow$ (2) (5)

05

---

(1) – (2) gives;

$|z-w|^2 - |1 - z\bar{w}|^2$

$= |z|^2 + |w|^2 - 1 - |z\bar{w}|^2$ (5)

$= -(1 - |w|^2 - |z|^2 + |z|^2\,|w|^2)$ (5)

$= -(1 - |z|^2)(1 - |w|^2)$ (5)

15

$|w| = 1,\ z \neq w$

$\Rightarrow |z-w|^2 - |1 - \bar{z}w|^2 = 0$ (5)

$\Rightarrow |z-w| = |1 - \bar{z}w|$

$\Rightarrow \dfrac{|z-w|}{|1-\bar{z}w|} = 1$ $\left[\begin{array}{l}\because z \neq w \\ \Rightarrow \bar{z}w \neq 1\end{array}\right]$

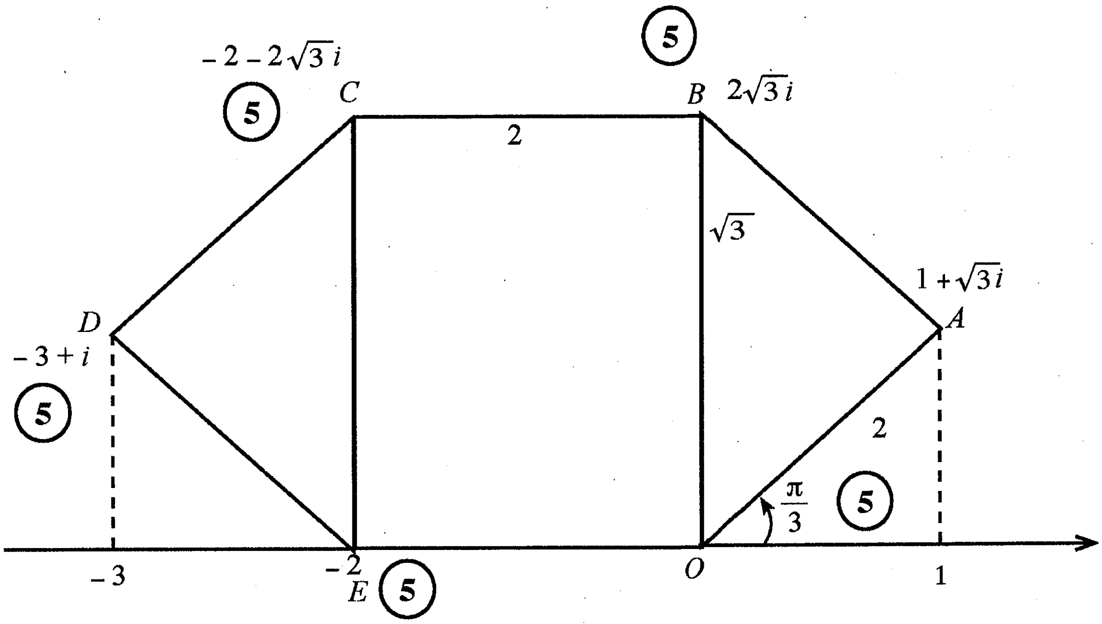
$\Rightarrow \left|\dfrac{z-w}{1-\bar{z}w}\right| = 1$ (5)

10

(*c*) $1+\sqrt{3}\,i \;=2\left\{\frac{1}{2}+i\frac{\sqrt{3}}{2}\right\}$ (5)

$=2\left\{\cos\frac{\pi}{3}+i\,\sin\frac{\pi}{3}\right\}.$ (5)

10



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14.(*a*) Let $f(x)=\dfrac{x(2x-3)}{(x-3)^2}$ for $x\neq 3$.

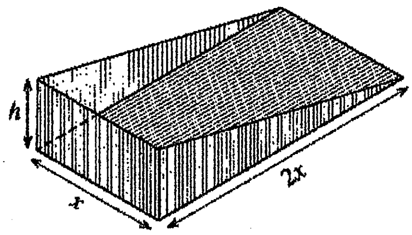
Show that $f'(x)$, the derivative of $f(x)$, is given by $f'(x)=\dfrac{9(1-x)}{(x-3)^3}$ for $x\neq 3$.

Hence, find the interval on which $f(x)$ is increasing and the intervals on which $f(x)$ is decreasing. Also find the coordinates of the turning point of $f(x)$.

Sketch the graph of $y=f(x)$ indicating the asymptotes, the turning point and the x-intercepts.

Using the graph, find all real values of x satisfying the inequality $\dfrac{1}{1+f(x)}\leq\dfrac{1}{3}$.

(*b*) The adjoining figure shows the portion of a dust pan without its handle. Its dimensions in centimetres, are shown in the figure. It is given that its volume x^2h cm³ is 4500 cm³. Its surface area S cm² is given by $S=2x^2+3xh$. Show that S is minimum when $x=15$.



a) For $x\neq 3$; $f(x)=\dfrac{x(2x-3)}{(x-3)^2}$

Then, $f'(x)=\dfrac{1}{(x-3)^2}\left[2x-3+2x\right]-\dfrac{2x(2x-3)}{(x-3)^3}$ (20)

$=\dfrac{(x-3)(4x-3)-2x(2x-3)}{(x-3)^3}$

$=\dfrac{4x^2-15x+9-4x^2+6x}{(x-3)^3}$

$=\dfrac{9(1-x)}{(x-3)^3}$ (5)

25

$f'(x)=0\Leftrightarrow x=1.$ (5)

	$-\infty<x<1$	$1<x<3$	$3<x<\infty$
Sign of $f'(x)$	(−)	(+)	(−)
$f(x)$ is	↘ Decreasing	↗ Increasing	↘ Decreasing
	(5)	(5)	(5)

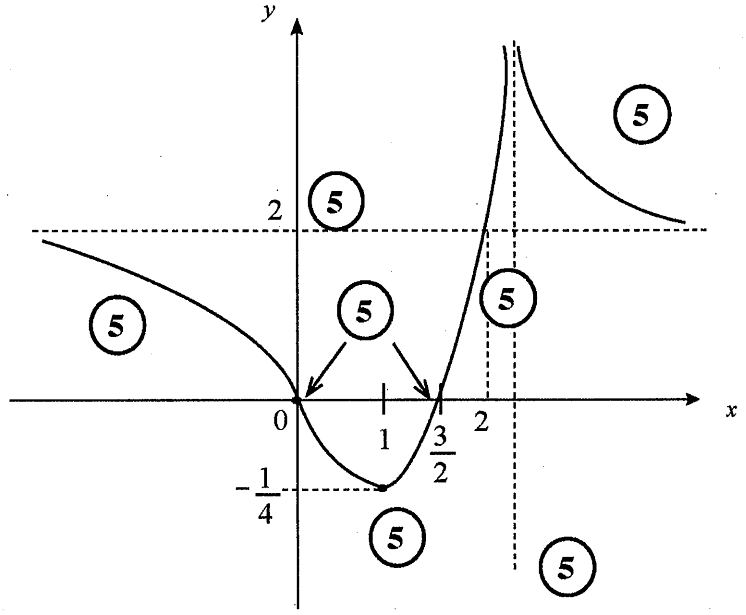
20

Turning point : $\left(1, -\frac{1}{4}\right)$ is a local minimum. (5)

05

Horizontal asymptote : $\lim_{x \to \pm\infty} f(x) = 2 \quad \therefore\ y = 2$ (5)

Vertical asymptote : $x = 3$. (5)



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$$\frac{1}{1+f(x)} \leq \frac{1}{3}.$$

Note that $1+f(x) > 0$.

$\therefore 3 \leq 1 + f(x)$.

$\therefore f(x) \geq 2$. (5)

$f(x) = 2 \Leftrightarrow x(2x-3) = 2(x-3)^2$. (5)

$\Leftrightarrow \quad 2x^2 - 3x = 2\{x^2 - 6x + 9)$

$\Leftrightarrow \quad x = 2$ (5)

The required values x are $2 \leq x \leq 3$ or $x > 3$.

05

(*b*) $x^2 h = 4500.$

Hence, $S = 2x^2 + 3xh$

$$= 2x^2 + 3x \cdot \frac{4500}{x^2} \quad \text{for } x > 0.$$ (5)

$$\therefore \frac{dS}{dx} = 4x - 3 \times 4500\left(\frac{1}{x^2}\right) = \frac{4(x^3 - 3375)}{x^2}.$$ (5)

$\frac{dS}{dx} = 0$ (5) $\Leftrightarrow x = 15.$ (5)

For $0 < x < 15$, $\frac{ds}{dr} < 0$ (5) and for $x > 15$, $\frac{ds}{dr} > 0.$ (5)

$\therefore$ S is minimum when $x = 15.$ (5)

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Confidential

NEW/OLD

Department of Examination - Sri Lanka

G.C.E. (A/L) Examination - 2020

10 - Combined Mathematics - II

NEW/OLD Syllabus

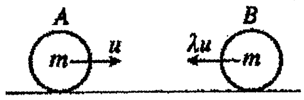
Marking Scheme

This document has been prepared for the use of Marking Examiners. Some changes would be made according to the views presented at the Chief Examiners' meeting.

Amendments to be included

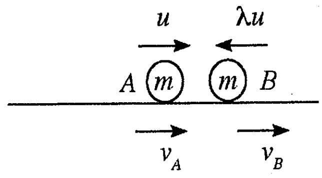
New Syllabus

1. Two particles A and B each of mass m, moving in the same straight line on a smooth horizontal floor, but in opposite directions collide directly. The velocities of A and B just before collision are u and λu, respectively. The coefficient of restitution between A and B is $\frac{1}{2}$.



Find the velocity of A just after collision and show that if $\lambda > \frac{1}{3}$, then the direction of motion of A is reversed.

For A and B, applying $\underline{I} = \Delta(m\underline{v})$, $\rightarrow$:



$(mv_A + mv_B) - (mu - m\lambda u) = 0$

$v_A + v_B = (1 - \lambda)u$ ——— (1) **(10)**

(5) if only one moment um is correct

Newton's Expermental law :

$v_B - v_A = \frac{1}{2}(u + \lambda u)$ ——— (2) **(5)**

(1) - (2) : $2v_A = u - \lambda u - \frac{1}{2}u - \frac{\lambda}{2}u$

$v_A = \frac{1}{4}(1 - 3\lambda)u$ **(5)**

If $\lambda > \frac{1}{3}$, then $v_A < 0$. **(5)**

$\therefore$ The direction of motion of A is reversed.

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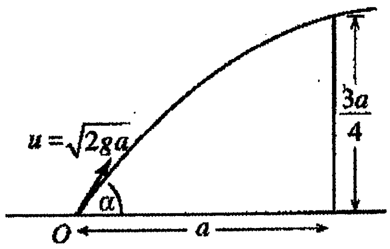
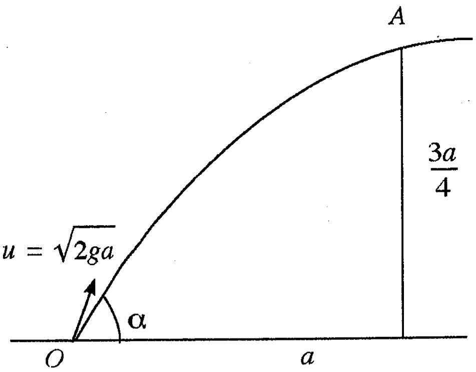
2. A particle is projected from a point O on a horizontal floor with initial velocity $u=\sqrt{2ga}$ and at an angle $\alpha\left(0<\alpha<\frac{\pi}{2}\right)$ to the horizontal. The particle just clears a vertical wall of height $\frac{3a}{4}$ located at a horizontal distance a from O.

Show that $\sec^2\alpha - 4\tan\alpha + 3 = 0$.

Hence, show that $\alpha = \tan^{-1}(2)$.





Let t be the time taken from O to A

Applying $S = ut + \frac{1}{2}at^2$:

$\rightarrow$ $a = u\cos\alpha\, t$ —— (1) (5)

$\uparrow$ $\frac{3a}{4} = u\sin\alpha\, t - \frac{1}{2}gt^2$ —— (2) (5)

(1) $\Rightarrow t = \frac{a}{u\cos\alpha}$

Now (2) $\Rightarrow \frac{3a}{4} = a\tan\alpha - \frac{1}{2}g\frac{a^2}{2\,ga\cos^2\alpha}$

$\Rightarrow \frac{3}{4} = \tan\alpha - \frac{1}{4}\sec^2\alpha$

$\Rightarrow \sec^2\alpha - 4\tan\alpha + 3 = 0$ (5)

$\Rightarrow (1+\sec^2\alpha) - 4\tan\alpha + 3 = 0$ (5)

$\tan^2\alpha - 4\tan\alpha + 4 = 0$

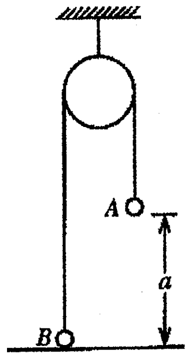
$\Rightarrow (\tan\alpha - 2)^2 = 0$

$\tan\alpha = 2$

$\therefore\ \alpha = \tan^{-1}(2)$. (5)

25

3. Two particles A and B, each of mass m, attached to the two ends of a light inextensible string which passes over a fixed smooth pulley are in equilibrium with the particle A at a height a from a horizontal floor and the particle B touching the floor, as shown in the figure. Now, the particle A is given an impulse mu vertically downwards. Find the velocity of the particle A just after the impulse. Write down the time taken by A to reach the floor.

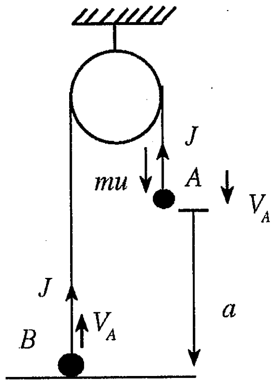


Applying $\underline{I} = \Delta(m\underline{v})$

(A) ↓ $mu - J = mV_A$ (5)

(B) ↑ $J = mV_A$ (5)

$\therefore \; V_A = \dfrac{u}{2}$ (5)



(5)

$T = \dfrac{a}{V_A} = \dfrac{2a}{u}$ (5)

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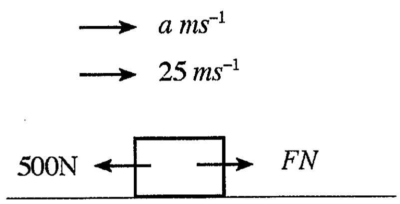
4. **A car of mass 1500 kg travels on a straight horizontal road against a constant resistance of magnitude 500 N. Find the acceleration of the car when the engine of the car is working at power 50 kW and the car is travelling with speed 25 m s^{-1}.**
At this instant, the engine of the car is turned off. Find the speed of the car after 50 seconds from the instant the engine was turned off.



Since the power = 50kW, we have

$50 \times 10^3 = F \times 25$ (5)

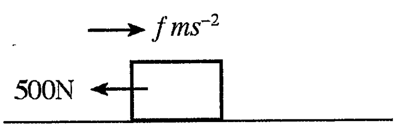
$\Rightarrow F = 2000$

Applying $\underline{F} = m\underline{a} \longrightarrow$

$F - 500 = 1500\, a$ (5)

$a = 1$ (5)

When the engine of the car is turned off,



$\underline{F} = m\underline{a} \longrightarrow$

$-500 = 1500 f$ (5)

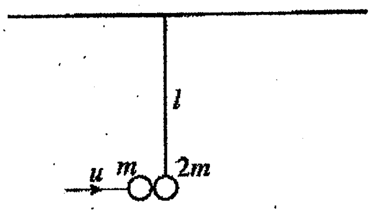
$f = -\dfrac{1}{3}$

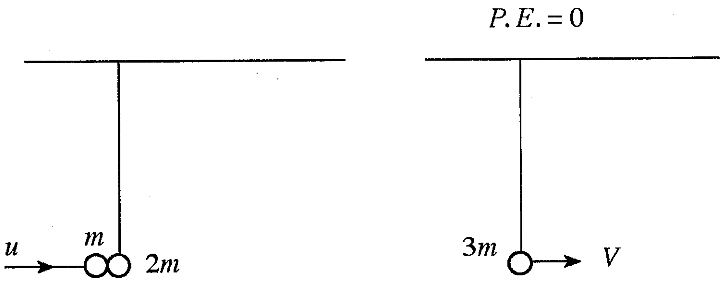
Applying $v = u + at \longrightarrow$ $v = 25 - \dfrac{1}{3} \times 50$

$v = \dfrac{25}{3}\ ms^{-1}$ (5)

25

5. A particle P of mass $2m$, hanging freely from a horizontal ceiling by a light inextensible string of length l, is in equilibrium. Another particle of mass m moving in a horizontal direction with velocity u collides with the particle P and coalesces to it. The string remains taut after the collision and the composite particle just reaches the ceiling. Show that $u = \sqrt{18gl}$.





Applying $\underline{I} = \Delta(m\underline{v})$

for m and $2m \longrightarrow 0 = 3mV - (mu)$ (5)

$V = \frac{u}{3}$ (5)

Applying the principle of conservation of energy for the composite particle:

$\frac{1}{2}(3m)V^2 - 3mgl = 0$ (10)

$V^2 = 2gl$

$\frac{u^2}{9} = 2gl$

$u = \sqrt{18gl}$ (5)

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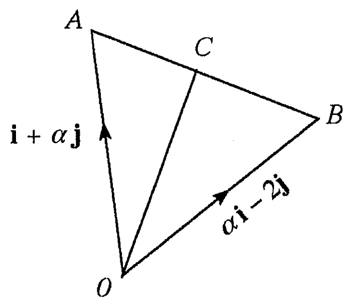
6. Let $\alpha > 0$ and in the usual notation, let $\mathbf{i}+\alpha\mathbf{j}$ and $\alpha\mathbf{i}-2\mathbf{j}$ be the position vectors of two points A and B, respectively, with respect to a fixed origin O. Also, let C be the point on AB such that $AC : CB = 1:2$. It is given that OC is perpendicular to AB. Find the value of α.



$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$$

$$= -(\mathbf{i} + \alpha\mathbf{j}) + (\alpha\mathbf{j} - z\mathbf{j}) \quad (5)$$

$$= (\alpha - 1)\,\mathbf{i} - (\alpha + 2)\,\mathbf{j}$$

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$$

$$= \overrightarrow{OA} + \frac{1}{3}\overrightarrow{AB} \quad (5)$$

$$= (\mathbf{i} + \alpha\mathbf{j}) + \frac{1}{3}[(\alpha - 1)\,\mathbf{i} - (\alpha + 2)\,\mathbf{j}\,] \quad (5)$$

$$= (\mathbf{i} + \alpha\mathbf{j}) + \frac{1}{3}[(\alpha - 1)\,\mathbf{i} - (\alpha + 1)\,\mathbf{j}\,]$$

$$= \frac{1}{3}[(\alpha + 2)\,\mathbf{i} + 2\,(\alpha - 1)\,\mathbf{j}\,]$$

$$\overrightarrow{OC} \perp \overrightarrow{AB} \Leftrightarrow \overrightarrow{OC} \cdot \overrightarrow{AB} = 0 \quad (5)$$

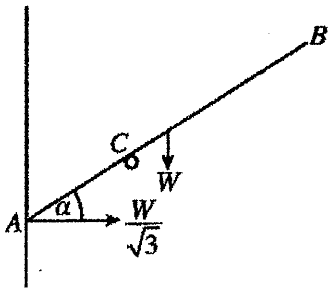
$$\Leftrightarrow (\alpha - 1)(\alpha + 2) - 2\,(\alpha + 2)(\alpha - 1) = 0$$

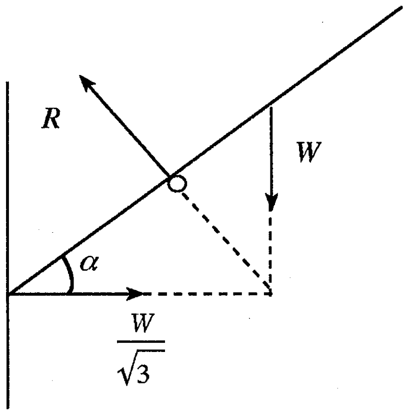
$$\Leftrightarrow (\alpha - 1)(\alpha + 2) = 0$$

$$\Leftrightarrow \alpha = 1 \quad (5) \quad (\because \alpha > 0)$$

25

7. A uniform rod ACB of length $2a$ and weight W is kept in equilibrium with the end A against a smooth vertical wall by a smooth peg placed at C, as shown in the figure. It is given that the reaction at A from the wall is $\frac{W}{\sqrt{3}}$. Show that the angle α that the rod makes with the horizontal is $\frac{\pi}{6}$.
Show also that $AC = \frac{3}{4}a$.





For the equilibrium of the rod:

$\rightarrow$ $R \sin\alpha = \frac{W}{\sqrt{3}}$ —— (1) **(5)**

$\uparrow$ $R \cos\alpha = W$ —— (2) **(5)**

$\frac{(1)}{(2)} \Rightarrow \tan\alpha = \frac{1}{\sqrt{3}}$

$\Rightarrow \alpha = \frac{\pi}{6}$ **(5)**

Now (1) $\Rightarrow R = \frac{2W}{\sqrt{3}}$

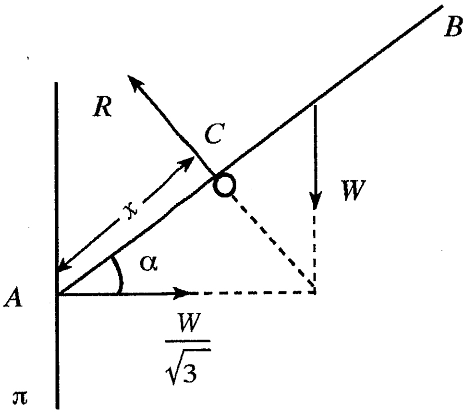
A ↶ $R \times AC = W \times a\cos\frac{\pi}{6}$ (or $Wa\cos\alpha$) **(5)**

$\frac{2W}{\sqrt{3}} \times AC = W \times a \times \frac{\sqrt{3}}{2}$

$AC = \frac{3}{4}a$ **(5)**

25

Aliter 1



AB ↗ $\frac{W}{\sqrt{3}} \cos\alpha = W \sin\alpha$ (10)

$\Rightarrow \tan\alpha = \frac{1}{\sqrt{3}}$

$\Rightarrow \alpha = \frac{\pi}{6}$ (5)

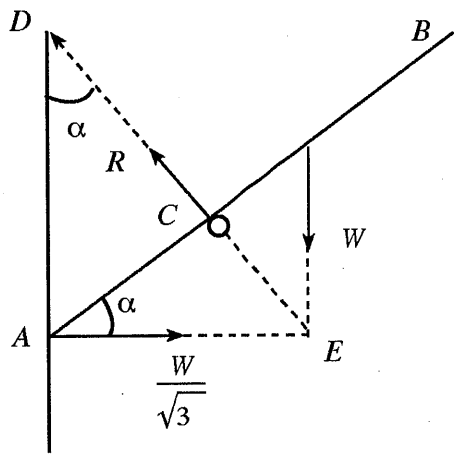
C ↶ $\frac{W}{\sqrt{3}} \times x \sin\frac{\pi}{6} = W \times (a - x) \cos\frac{\pi}{6}$ (5)

$\frac{1}{\sqrt{3}} \times x \times \frac{1}{2} = (a - x)\frac{\sqrt{3}}{2}$

$x = 3(a - x)$

$x = \frac{3}{4} a$ (5)

Aliter 2



Δ ADE is a force Δ (5)

$\frac{\frac{W}{\sqrt{3}}}{AE} = \frac{W}{AD}$

$\frac{AE}{AD} = \frac{1}{\sqrt{3}}$ (5)

$\Rightarrow \tan\alpha = \frac{1}{\sqrt{3}}$

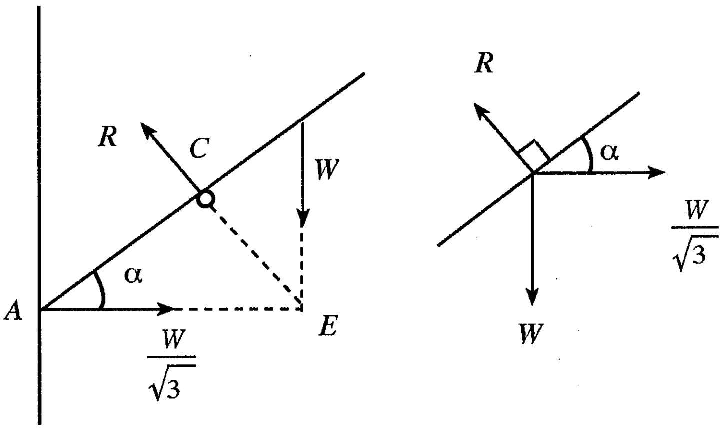
$\Rightarrow \alpha = \frac{\pi}{6}$ (5)

$\therefore AE = a\cos\frac{\pi}{6} = \frac{a\sqrt{3}}{2}$ (5)

$AC = AE \cos\frac{\pi}{6} = \frac{a\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}$

$= \frac{3}{4} a$ (5)

Aliter 3



By Lami's Rule:

$$\frac{W}{\sin(\frac{\pi}{2}+\alpha)} = \frac{\frac{W}{\sqrt{3}}}{\sin(\pi-\alpha)} \quad (5)$$

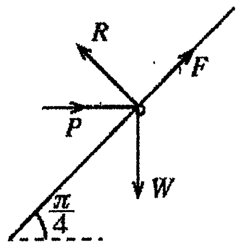
$$\frac{1}{\cos\alpha} = \frac{1}{\sqrt{3}\sin\alpha} \quad (5)$$

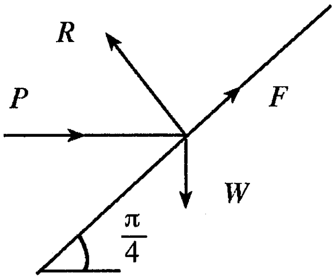
$$\Rightarrow \tan\alpha = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \alpha = \frac{\pi}{6} \quad (5)$$

$\overset{\frown}{C}$ ***OR*** from Δ ACE we get $AC = \frac{3}{4}a$. (5) + (5)

8. A small bead of weight W is threaded to a fixed rough straight wire inclined at an angle $\frac{\pi}{4}$ to the horizontal. The bead is kept in equilibrium by a horizontal force of magnitude P as shown in the figure. The coefficient of friction between the bead and the wire is $\frac{1}{2}$.
Obtain equations sufficient to determine the frictional force F and the normal reaction R on the bead, in terms of P and W.
It is given that $\frac{F}{R} = \frac{W-P}{W+P}$. Show that $\frac{W}{3} \le P \le 3W$.





$$F = \frac{W-P}{W+P}$$

For the equilibrium of the bead:

 $F - \frac{W}{\sqrt{2}} + \frac{P}{\sqrt{2}} = 0.$ (5) (or with $\cos\frac{\pi}{4}$, $\sin\frac{\pi}{4}$)

 $R - \frac{W}{\sqrt{2}} - \frac{P}{\sqrt{2}} = 0.$ (5) (or with $\cos\frac{\pi}{4}$, $\sin\frac{\pi}{4}$)

OR ↑ →

$$\mu \ge \frac{|F|}{R}$$

$$\frac{1}{2} \ge \frac{|W-P|}{W+P} \quad (10)$$

Only (5) without the absolute value

$$|W-P| \le \frac{1}{2}(W+P)$$

$$-\frac{1}{2}(W+P) \le W-P \le \frac{1}{2}(W+P)$$

$$-W-P \le 2W-2P \le W+P$$

$$\frac{W}{3} \le P \le 3W \quad (5)$$

25

9. Let A and B be two events of a sample space Ω. In the usual notation, it is given that $P(A)=\frac{3}{5}$, $P(B|A)=\frac{1}{4}$ and $P(A\cup B)=\frac{4}{5}$. Find $P(B)$.
Show that the events A and B are not independent.

$$P(B|A) = \frac{P(B\cap A)}{P(A)}$$

$$\Rightarrow \quad P(A\cap B) = \frac{3}{5}\times\frac{1}{4}=\frac{3}{20} \quad ⑤$$

$$\Rightarrow \quad P(A\cup B) = P(A)+P(B)-P(A\cap B) \quad ⑤$$

$$\frac{4}{5} = \frac{3}{5} + P(B) - \frac{3}{20}$$

$$P(B) = \frac{16}{20} - \frac{12}{20} + \frac{3}{20} = \frac{7}{20} \quad ⑤$$

$$P(A).P(B) = \frac{3}{5}\times\frac{7}{20} = \frac{21}{100} \quad ⑤$$

$$\therefore \quad P(A\cap B) \neq P(A).P(B) \quad ⑤$$

$\therefore$ A and B are not independent.

25

10. A set of 5 observations of positive integers, each less than or equal to 10, has mean, media and mode each equals to 6. The range of the observations is 9. Find these five observations.

Mode $= 6 \Rightarrow$ At least two of the numbers must be 6, 6 (5)

Range $= 9$ and the numbers are positive intergers ≤ 10, we have the smallest is 1 and the largest is 10. (5)

Since the median is 6, the numbers

must be $1, a, 6, 6, 10$ or $1, 6, 6, a, 10$. (5)

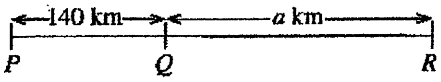
Mean $= 6$ gives $\dfrac{a + 23}{5} = 6.$ (5)

$\therefore \; a = 7$ (5)

$\therefore$ The numbers are 1, 6, 6, 7, 10.

25

11. (*a*) Three railway stations P, Q and R located in a straight line such that $PQ = 140$ km and $QR = a$ km, as shown in the figure. At time $t = 0$, a train A starts from rest at P and moves towards Q with constant acceleration f km h^{-2} for half an hour and maintains the velocity it had at time $t = \frac{1}{2}$ h for three hours. Then it moves with constant retardation f km h^{-2} and comes to rest at Q. At time $t = 1$ h, another train B starts from rest at R and moves towards Q with constant acceleration $2f$ km h^{-2} for T hours and then with a constant retardation f km h^{-2} and comes to rest at Q. Both trains come to rest at the same instant. Sketch velocity-time graphs for the motions of A and B in the same diagram.



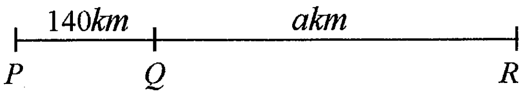
Hence or otherwise, show that $f = 80$ and find the values of T and a. **20**

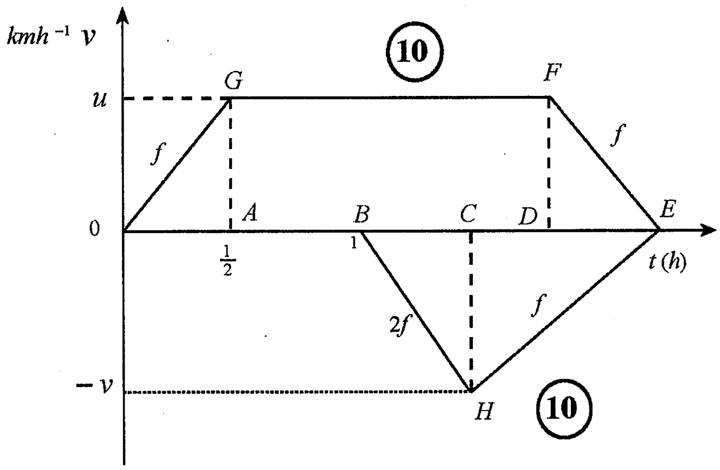
(*b*) A ship is sailing due west with uniform speed u relative to earth and a boat is sailing in a straight line path with uniform speed $\frac{u}{2}$ relative to earth. At a certain instant, the ship is at a distance d at an angle $\frac{\pi}{3}$ east of north from the boat.

(i) If the boat is sailing relative to earth in the direction making an angle $\frac{\pi}{6}$ west of north, show that the boat can intercept the ship and that the time taken by the boat to intercept the ship is $\frac{2d}{\sqrt{3}u}$.

(ii) If the boat is sailing relative to earth in the direction making an angle $\frac{\pi}{6}$ east of north, show that the speed of the boat relative to the ship is $\frac{\sqrt{7}u}{2}$ and that the shortest distance between the ship and the boat is $\frac{d}{2\sqrt{7}}$.

(*a*)





20

<u>$\Delta\ OAG$</u>

$$f = \frac{u}{\frac{1}{2}}$$

$\therefore\ f = 2u$

$\Delta\ OAG \equiv \Delta\ DEF$

$\therefore DE = \frac{1}{2}$ (5)

Area of the trapezium $OEFG = 140$ (5)

$\frac{1}{2}(4+3)\,u = 140$ (5)

$\therefore\ u = 40$

$\therefore\ f = 80.$ (5)

25

<u>$\Delta\ BHC$</u>

$2f = \frac{V}{T} \Rightarrow 160 = \frac{V}{T}$ (5)

<u>$\Delta\ ECH$</u>

$f = \frac{V}{CE} \Rightarrow 80 = \frac{V}{CE}$ (5)

$\therefore CE = 2T$ (5)

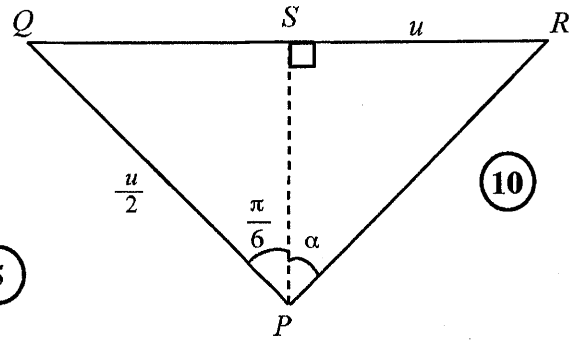
$\therefore\ 3T = 3$ and $T = 1.$ (5) Also $V = 160.$

a = Area of $\Delta\ BHE = \frac{1}{2} \times 3 \times 160$

$= 240$ (5)

30

(b) $\mathbf{V}(S, E) = \leftarrow u$ (5)



(i) $\mathbf{V}(B, E) = \frac{u}{2}$ ↖ $\frac{\pi}{6}$ (5)

$\mathbf{V}(B, S) = \mathbf{V}(B, E) + \mathbf{V}(E, S)$ (5)

$= \overrightarrow{PQ} + \overrightarrow{QR}$

$= \overrightarrow{PR}$

$QS = \frac{u}{2} \sin \frac{\pi}{6} = \frac{u}{4}$

$\therefore\ SR = \frac{3u}{4}$

$SP = \frac{u}{2} \cos \frac{\pi}{6} = \frac{\sqrt{3}u}{4}$

$\tan\alpha = \frac{SR}{SP} = \frac{3u}{4} \times \frac{4}{\sqrt{3}u} = \sqrt{3}$ (10)

$\therefore\ \alpha = \frac{\pi}{3}$ (5)

$\therefore$ Boat can intercept the ship.

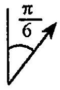
40

$Q\hat{P}R = \frac{\pi}{2}$

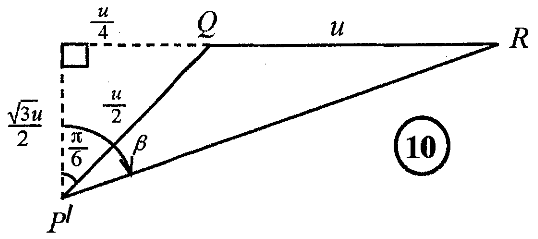
$\therefore\ PR = \frac{\sqrt{3}\,u}{2}$ (5)

$t = \frac{d}{PR} = \frac{2d}{\sqrt{3}\,u}$ (5)

10

(ii) $\mathbf{V}(B, E) =$ 

(5)



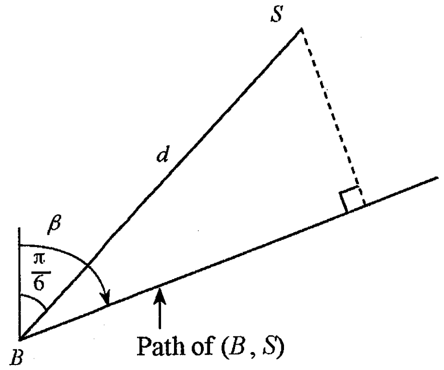
$\mathbf{V}(B, S) = \mathbf{V}(B, E) + \mathbf{V}(E, S)$

$= \overrightarrow{P'Q} + \overrightarrow{QR}$

$= \overrightarrow{P'R}$

From the velocity triangle,

$\sin\beta = \frac{5}{2\sqrt{7}} \quad \cos\beta = \frac{\sqrt{3}}{2\sqrt{7}}$



Shortest distance $= d \sin(\beta - \frac{\pi}{3})$ (5)

$= d(\sin\beta \cos\frac{\pi}{3} - \cos\beta \sin\frac{\pi}{3})$ (5)

$= d(\frac{5}{4\sqrt{7}} - \frac{3}{4\sqrt{7}})$

$= \frac{d}{2\sqrt{7}}$ (5)

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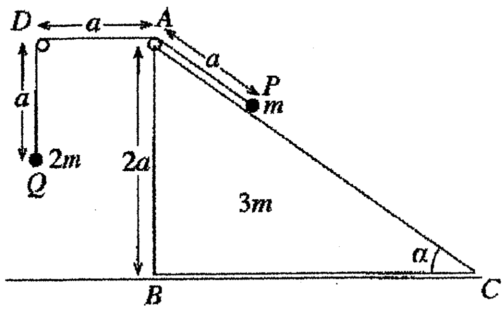
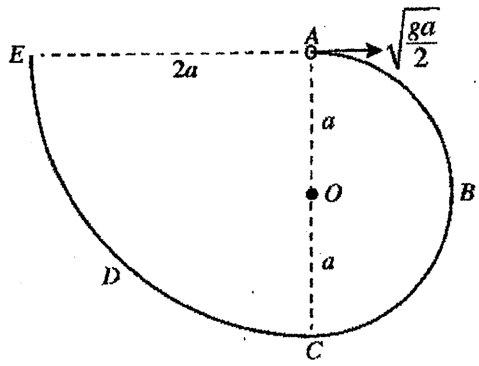
12.(*a*) The triangle ABC in the figure is the vertical cross-section through the centre of gravity of a smooth uniform wedge of mass $3m$ with $A\hat{C}B = \alpha$, $A\hat{B}C = \frac{\pi}{2}$ and $AB = 2a$ such that the face containing BC is placed on a smooth horizontal floor. The line AC is a line of greatest slope of the face containing it. The point D is a fixed point in the plane of ABC such that AD is horizontal. Two particles P and Q of masses m and $2m$, respectively are attached to the two ends of a light inextensible string of length $3a$ passing over smooth small pulleys fixed at A and D. The system is released from rest with the particle P held on AC and the particle Q hanging freely such that $AP = AD = DQ = a$, as shown in the figure. Obtain equations sufficient to determine the time taken by the particle Q to reach the floor.



(*b*) A smooth thin wire $ABCDE$ is fixed in a vertical plane, as shown in the figure. The portion ABC is a semicircle with centre O and radius a, and the portion CDE is a quarter of a circle with centre A and radius $2a$. The points A and C lie on the vertical line through O and the line AE is horizontal. A small smooth bead P of mass m is placed at A and is given a velocity $\sqrt{\frac{ga}{2}}$ horizontally, and begins to move along the wire.

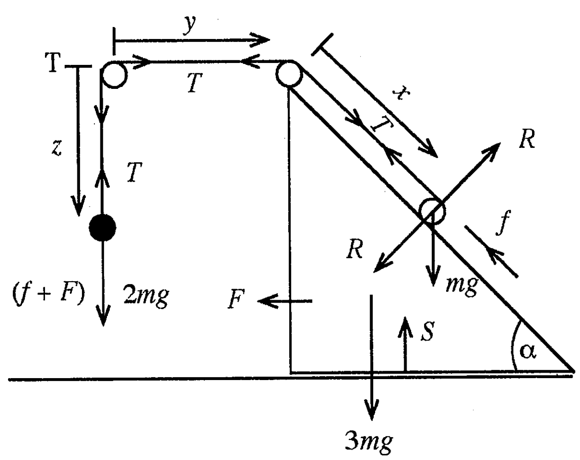


Show that the speed v of the bead P when $\overrightarrow{OP}$ makes an angle θ $(0 \leq \theta \leq \pi)$ with $\overrightarrow{OA}$ is given by $v^2 = \frac{ga}{2}(5 - 4\cos\theta)$.

Find the reaction on the bead P from the wire at the above position and show that it changes its direction when the bead P passes the point $\theta = \cos^{-1}\left(\frac{5}{6}\right)$.

Write down the velocity of the bead P just before it leaves the wire at E and find the reaction on the bead P from the wire at that instant.

(*a*)



Forces (15)

Accelerations (20)

$\ddot{z} = -\ddot{x} - \ddot{y}$

$= f + F$

Applying $F = ma$

For (2m) $\downarrow$ $2mg - T = 2m(f + F)$ (10)

For (m) $\nwarrow$ $T - mg\sin\alpha = m(f + F\cos\alpha)$ (10)

For ((m) and (3m)) $\leftarrow$ $T = 3mF + m(F + f\cos\alpha)$ (15)

$a = \frac{1}{2}(f + F)t^2$ (10)

80

(b)

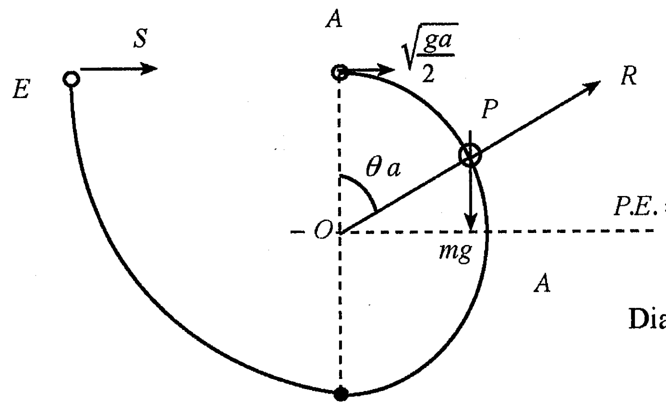


Diagram (10)

By the conservation of Energy,

P.E. + K.E. + equation (5) (5) (5)

$\frac{1}{2}mv^2 + mga\cos\theta = \frac{1}{2}m\left(\frac{ga}{2}\right) + mga$

$2v^2 + 4ga\cos\theta = 5ga$

$v^2 = \frac{ga}{2}(5 - 4\cos\theta)$ (5)

30

For circular motion, applying $\underline{F} = m\underline{a}$ $\nearrow$

$R - mg\cos\theta = -m\frac{V^2}{a}$ (10)

$R = mg\cos\theta - \frac{mg}{2}(5 - 4\cos\theta)$ (5)

$= \frac{mg}{2}(6\cos\theta - 5)$

$0 < \theta < \alpha$; $R > 0$ where $\cos\alpha = \frac{5}{6}$ (5)

$\alpha < \theta < \pi$; $R < 0$

Hence the reaction changes its direction when bead passes the point $\theta = \cos^{-1}\left(\frac{5}{6}\right)$.

20

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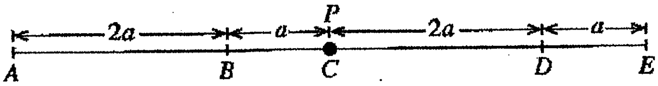
13. The points A, B, C, D and E lie on a straight line in that order, on a smooth horizontal table such that $AB = 2a$, $BC = a$, $CD = 2a$ and $DE = a$, as shown in the figure.



One end of a light elastic string of natural length $2a$ and modulus of elasticity kmg is attached to the point A and the other end to a particle P of mass m. One end of another light elastic string of natural length a and modulus of elasticity mg is attached to the point E and the other end to the particle P. When the particle P is held at C and released, it stays in equilibrium. Find the value of k.

Now, the string AP is pulled until the particle P reaches the point D and released from rest. Show that the equation of motion of P from D to B is given by $\ddot{x}+\frac{3g}{a}x=0$, where $CP = x$.

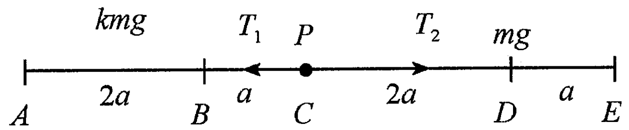
Using the formula $\dot{x}^2 = \frac{3g}{a}(c^2 - x^2)$, where c is the amplitude, show that the velocity of particle P when it reaches B is $3\sqrt{ga}$.

An impulse is given to the particle P when it reaches B so that the velocity of P just after the impulse is $\sqrt{ag}$ in the direction of $\overrightarrow{BA}$.

Show that the equation of motion of P after passing B until it comes to instantaneous rest is given by $\ddot{y}+\frac{g}{a}y=0$, where $DP = y$.

Show that the total time taken by the particle P, started at D, to reach B for the second time is $2\sqrt{\frac{a}{g}}\left(\frac{\pi}{3\sqrt{3}}+\cos^{-1}\left(\frac{3}{\sqrt{10}}\right)\right)$.

13.



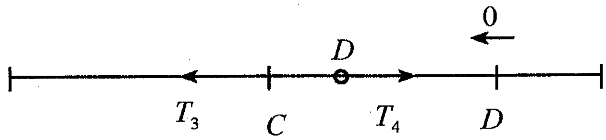
P is at equilibrium at C.

$\therefore\ T_1 - T_2 = 0$ (5)

$\Leftrightarrow kmg \cdot \frac{a}{2a} = mg \cdot \frac{2a}{a}$ (10)

$\Leftrightarrow k = 4$ (5)

20



For (P) → $\underline{F} = m\underline{a}$

$-T_3 + T_4 = m\ddot{x}$ (5)

$-4mg.\dfrac{(a+x)}{2a} + mg.\dfrac{(2a-x)}{a} = m\ddot{x}$ (15)

$\dfrac{g}{a}\left\{-2a-2x+2a-x\right\} = \ddot{x}$

$\ddot{x} = \dfrac{-3g}{a}x$ (5)

$\therefore\ \ddot{x} + \dfrac{3g}{a}x = 0$

This is valid for $-a \le x \le 2a$ **25**

The centre for this S.H.M. is C and $\dot{x} = 0$ when $x = 2a$.

(5)

$\therefore$ Amplitude of this S.H.M. is $2a$. (5)

$\therefore\ \dot{x}^2 = \dfrac{3g}{a}(4a^2 - x^2)$ (5)

Let v be the speed at B $(x = -a)$.

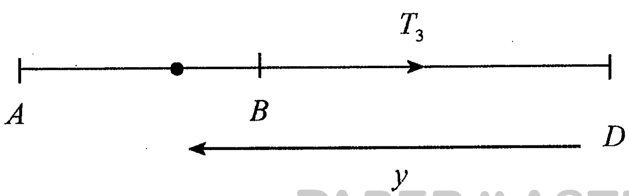
Then $v^2 = \dfrac{3g}{a}(4a^2 - a^2)$ (5)

$= 9ga$

$v = 3\sqrt{ga}$

$\therefore$ velocity when P reaches B for the first time is $3\sqrt{ga}$ ⟵ . (5) **25**

Due to the impulse, velocity just after impulse is $\sqrt{ga}$.



$-T_3 = m\ddot{y}$ (5)

$-mg\frac{y}{a} = m\ddot{y}$ (5)

$\therefore \ \ddot{y} = -\frac{g}{a}y$

or $\ddot{y} + \frac{g}{a}y = 0$ (5)

15

The centre of this S.H.M. is D. (5)

Let c be the amplitude.

$\dot{y} = \frac{g}{a}(c^2 - y^2)$

$\dot{y} = \sqrt{ga}$ when $y = 3a$ (5)

$ga = \frac{g}{a}(c^2 - 9a^2)$ (5)

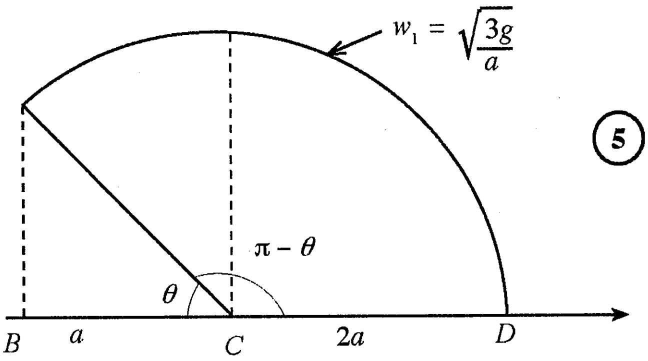
$c^2 = 10a^2$

$c = \sqrt{10}\,a$ (5)

Since $3a < \underbrace{\sqrt{10}\,a}_{c} < 5a$, the particle P will come to instantaneous rest at a point F between B and A.

20

Let $\tau_1 =$ Time taken from D to B.



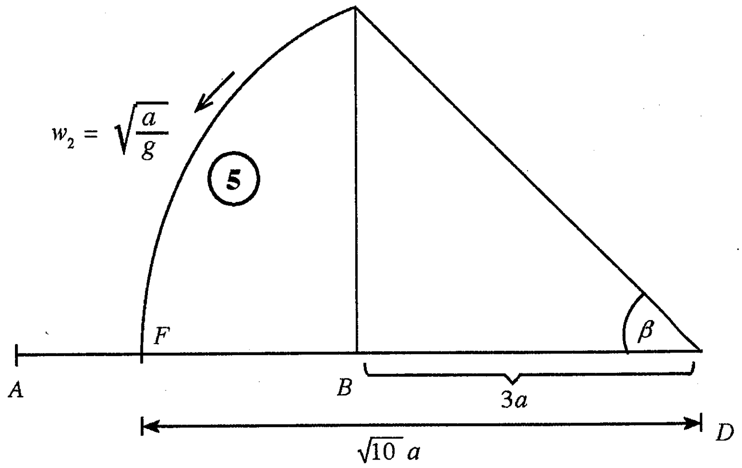
(5)

$\sqrt{\frac{3g}{a}}\,\tau_1 = \pi - \theta,$ (5) where $\cos\theta = \frac{a}{2a}$

$\theta = \frac{\pi}{3}$ (5)

$$\tau_1 = \sqrt{\frac{g}{3g}} \times \frac{2\pi}{3}$$

$$= \frac{2\pi}{3\sqrt{3}}\sqrt{\frac{a}{g}} \cdot \quad (5)$$



Let τ_2 = Time taken from B to F.

$\sqrt{\frac{a}{g}}\,\tau_2 = \beta$ (5)

$\cos\beta = \frac{3a}{\sqrt{10}\,a}$

$\therefore \tau_2 = \sqrt{\frac{a}{g}}\cos^{-1}\left(\frac{3}{\sqrt{10}}\right)$ (5)

$\beta = \cos^{-1}\left(\frac{3}{\sqrt{10}}\right)$

Let τ_3 = Time taken from F to B (Coming to B for the 2nd time)

$\tau_3 = \tau_2$

$\therefore$ The required time $= \tau_1 + 2\tau_2$ (5)

$$= 2\sqrt{\frac{a}{g}}\left\{\frac{\pi}{3\sqrt{3}} + \cos^{-1}\left(\frac{3}{\sqrt{10}}\right)\right\} \quad (5)$$

45

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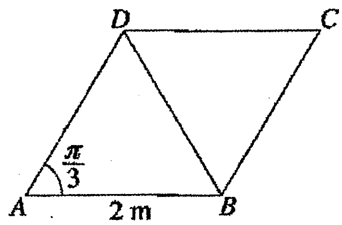
14. (*a*) Let **a** and **b** be two **unit vectors**.
The position vectors of three points A, B and C with respect to an origin O, are $12\mathbf{a}$, $18\mathbf{b}$ and $10\mathbf{a} + 3\mathbf{b}$ respectively. Express $\overrightarrow{AC}$ and $\overrightarrow{CB}$ in terms of **a** and **b**.
Deduce that A, B and C are collinear and find $AC : CB$.
It is given that $OC = \sqrt{139}$. Show that $A\hat{O}B = \frac{\pi}{3}$.

(*b*) Let $ABCD$ be a rhombus with $AB = 2$ m and $B\hat{A}D = \frac{\pi}{3}$. Forces of magnitude 10 N, 2 N, 6 N, P N and Q N act along AD, BA, BD, DC and CB respectively, in the directions indicated by the order of the letters. It is given that the resultant force is of magnitude 10 N and its direction is in the direction parallel to BC in the sense from B to C. Find the values of P and Q. Also, find the distance from A to the point where the line of action of the resultant force meets BA produced.



Now, a couple of moment M Nm acting in the counterclockwise sense and two forces, each of magnitude F N acting along CB and DC in the directions indicated by the order of the letters, are added to the system so that the resultant force passes through the points A and C. Find the values of F and M.

(*a*)

$$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$$
$$= \overrightarrow{OC} - \overrightarrow{OA} \quad (5)$$
$$= 10\mathbf{a} + 3\mathbf{b} - 12\mathbf{a}$$
$$= -2\mathbf{a} + 3\mathbf{b} \quad (5)$$

$$\overrightarrow{CB} = \overrightarrow{OB} - \overrightarrow{OC} \quad (5)$$
$$= 18\mathbf{b} - (10\mathbf{a} + 3\mathbf{b}) = -10\mathbf{a} + 15\mathbf{b} \quad (5)$$

20

$$\overrightarrow{CB} = 5\overrightarrow{AC} \quad (5)$$

$\therefore A, B$ and C are collinear (5)

and $AC : CB = 1 : 5$ (5)

15

$OC = \sqrt{139} \Rightarrow \overrightarrow{OC} \cdot \overrightarrow{OC} = 139$ (5)

$(10\mathbf{a} + 3\mathbf{b}) \cdot (10\mathbf{a} + 3\mathbf{b}) = 139$ (5)

$100|\mathbf{a}|^2 + 60\mathbf{a} \cdot \mathbf{b} + 9|\mathbf{b}|^2 = 139$ (5)

$60\mathbf{a} \cdot \mathbf{b} = 30$

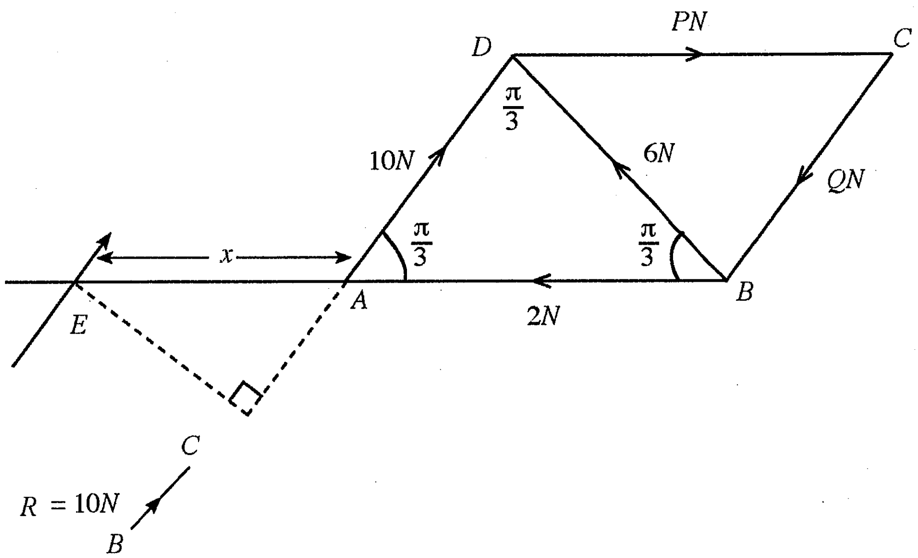
$\mathbf{a} \cdot \mathbf{b} = \frac{1}{2}$ (5)

$|\mathbf{a}|\ |\mathbf{b}| \cos A\hat{O}B = \frac{1}{2}$ (5)

$\therefore A\hat{O}B = \frac{\pi}{3}$ (5)

30

(*b*)



(10)

$\uparrow \quad 10 \sin \frac{\pi}{3} = 10 \sin \frac{\pi}{3} - Q \sin \frac{\pi}{3} - 6 \sin \frac{\pi}{3}$ (10)

$\therefore Q = 6$ (5)

$\rightarrow \quad 10 \cos \frac{\pi}{3} = P - 2 - 6 \cos \frac{\pi}{3} - 6 \cos \frac{\pi}{3} + 10 \cos \frac{\pi}{3}$ (10)

$\therefore P = 8$ (5)

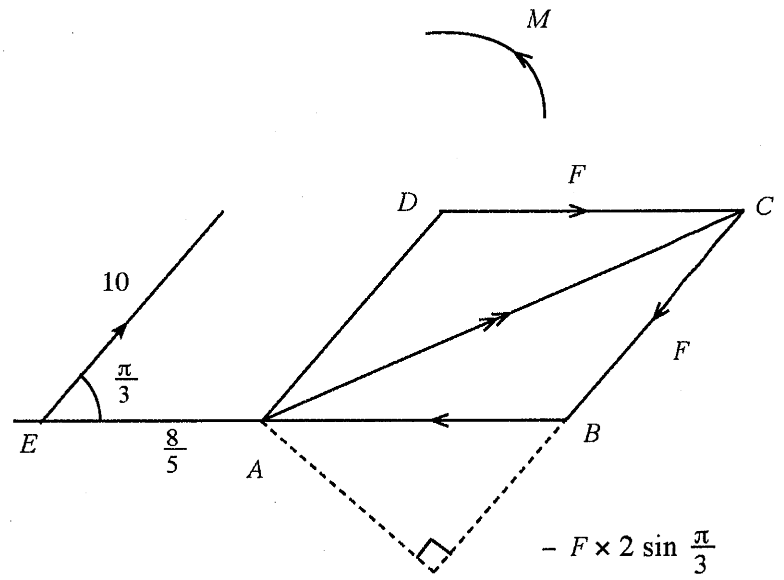
40

$E \curvearrowright$ $10x \sin\frac{\pi}{3} - 6x\ (2+x)\sin\frac{\pi}{3} - 8x2\sin\frac{\pi}{3} + 6\,(2+x)\sin\frac{\pi}{3} = 0$ (10)

$10x\,\frac{\sqrt{3}}{2} = 8\sqrt{3}$

$x = \frac{8}{5}\ m$ (5)

15



$A \curvearrowright$ $-10\times\frac{8}{5}\sin\frac{\pi}{3} + M - F\times\ 2\sin\frac{\pi}{3} = 0$ (10)

$M = F\times 2\sqrt{3} + 8\sqrt{3}$ (5)

$C \curvearrowright$ $M - 10\,(2+\frac{8}{5})\ \sin\frac{\pi}{3} = 0$ (5)

$M = 10\times\frac{18}{5}\times\frac{\sqrt{3}}{2}$

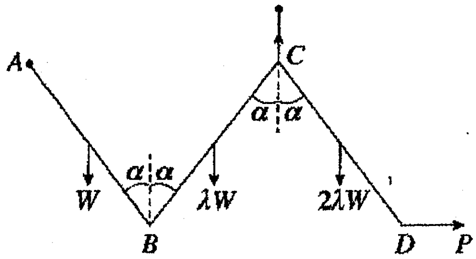
$= 18\sqrt{3}$ (5)

$F = \dfrac{18\sqrt{3} - 8\sqrt{3}}{2\sqrt{3}} = 5.$ (5)

30

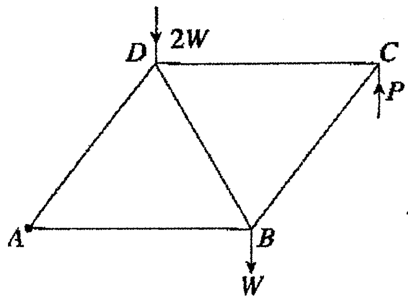
---

15. (*a*) Three uniform rods AB, BC and CD, each of length $2a$ are smoothly joined at the ends B and C. The weights of the rods AB, BC and CD are W, λW and $2\lambda W$, respectively. The end A is smoothly hinged to a fixed point. The rods are kept in equilibrium in a vertical plane by a light inextensible string attached to the joint C and to a fixed point vertically above C and by a horizontal force P applied to the end D such that A and C are at the same horizontal level and each of the rods making an angle α with the vertical, as shown in the figure. Show that $\lambda = \frac{1}{3}$.



Show also that the horizontal and vertical components of the force exerted on AB by CB at B are $\frac{W}{3}\tan\alpha$ and $\frac{W}{6}$, respectively.

(*b*) The framework shown in the adjoining figure is made from light rods AB, BC, CD, DA and BD, each of length $2a$, freely jointed at A, B, C and D. There are loads of W and $2W$ at B and D, respectively. The framework is smoothly hinged at A to a fixed point and kept in equilibrium with AB horizontal by a vertical force P applied to it at C, as shown in the figure. Find the value of P in terms of W.

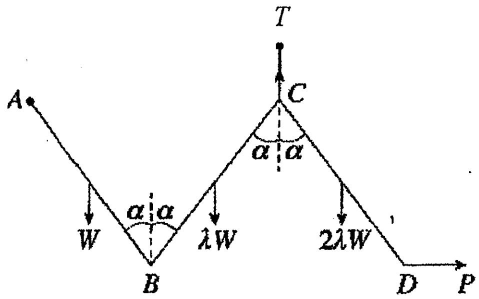


Draw a stress diagram using Bow's notation and **hence**, find the stresses in the rods stating whether they are tensions or thrusts.

(*a*)

Taking moments :

about C for CD



$\overset{\curvearrowright}{C}$ $2\lambda Wa\sin\alpha - P\,2a\cos\alpha = 0$ (5)

$\therefore\ P = \lambda W\tan\alpha$ (5)

about B for BC and CD

$\overset{\curvearrowright}{C}$ $\lambda Wa\sin\alpha - T\,2a\sin\alpha + 2\lambda W\,3a\sin\alpha = 0$ (10)

$\therefore\ T = \frac{7}{2}\lambda W$ (5)

about A for AB, BC and CD

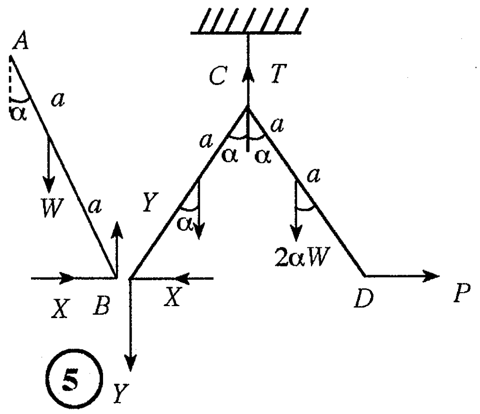
$A\curvearrowright$ $Wa\sin\alpha + \lambda W\,3a\sin\alpha - T\,4a\sin\alpha + 2\lambda W\,5a\sin\alpha - P\,2a\cos\alpha = 0$ (10)

$W\sin\alpha + 13\lambda W\sin\alpha - 14\lambda W\sin\alpha - \lambda W\tan\alpha\; 2\cos\alpha = 0$ (5)

$1 - \lambda - 2\lambda = 0$

$\Rightarrow \lambda = \frac{1}{3}$ (5)

45



For BC and CD

$\uparrow$ $Y + 3\lambda W - T = 0$

$\therefore Y = \frac{7}{2}\lambda W - 3\lambda W$ (5)

$= \frac{\lambda W}{2}$

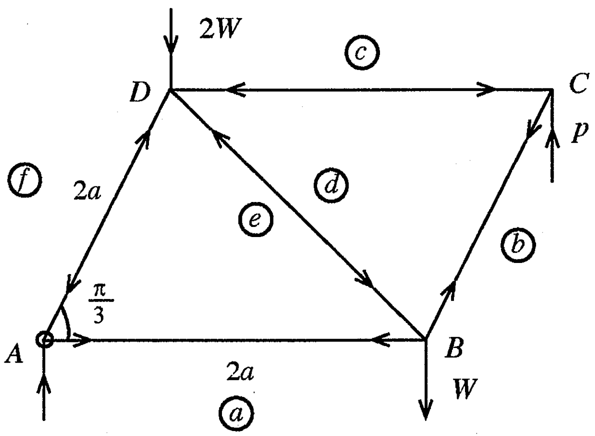
$= \frac{W}{6}$

$\leftarrow$ $X - P = 0$

$\therefore X = \frac{1}{3}W\tan\alpha$ (5)

15

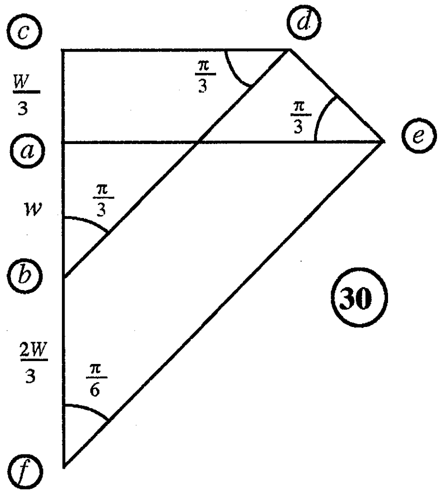
(b)



$\overset{\curvearrowright}{A}$ $2Wa + W2a - P3a = 0$

$P = \dfrac{4W}{3}$ (10)

10



(10 for each joints)

(30)

30

Rod	Tension	Thrust
AB	$\dfrac{5\sqrt{3}\,W}{9}$	–
BC	$\dfrac{8\sqrt{3}\,W}{9}$	–
CD	–	$\dfrac{4\sqrt{3}\,W}{9}$
DA	–	$\dfrac{10\sqrt{3}\,W}{9}$
BD	–	$\dfrac{2\sqrt{3}\,W}{9}$

(10) (10) (10) (10) (10)

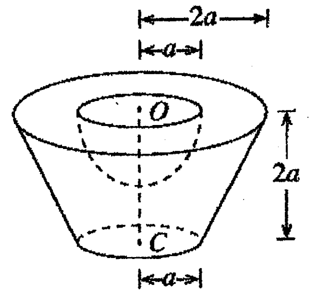
50

16. Show that the centre of mass of

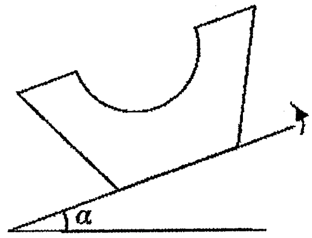
(i) a uniform solid right circular cone of base radius r and height h is at a distance $\frac{h}{4}$ from the centre of the base,

(ii) a uniform solid hemisphere of radius r is at a distance $\frac{3r}{8}$ from its centre.

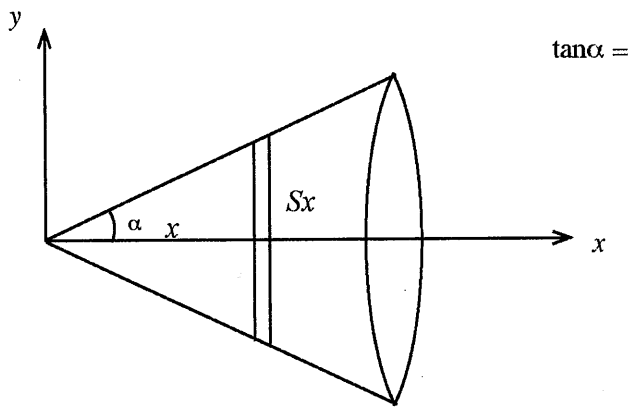
The adjoining figure shows a mortar S made by removing a solid hemisphere from a frustum of a solid uniform right circular cone having base radius $2a$ and height $4a$. The radius and the centre of the upper circular face of the frustum are $2a$ and O, respectively, and those for the lower circular face are a and C, respectively. The height of the frustum is $2a$. The radius and the centre of the removed solid hemisphere are a and O, respectively. Show that the centre of mass of mortar S lies at a distance $\frac{41}{48}a$ from O.



Mortar S is placed on a rough horizontal plane with its lower circular face touching the plane. Now, the plane is tilted upwards slowly. The coefficient of friction between the mortar and the plane is 0.9. Show that if $\alpha < \tan^{-1}(0.9)$, then the mortar stays in equilibrium, where α is the inclination of the plane to the horizontal.



(i) Uniform solid right circular cone



$\tan\alpha = \frac{r}{h}$

By symmetry, the centre of mass lies on the x - axis. (5)

$Sx = \pi (x\tan\alpha)^2\, Sx\, \rho$, where ρ is the density.

$$\bar{x} = \frac{\int_0^h \pi\tan^2\alpha\,\rho x^2 . x\,dx}{\int_0^h \pi\tan^2\alpha\,\rho x^2\,dx} \quad (5)\ (5)$$

$$= \frac{\left.\frac{x^4}{4}\right|_0^h}{\left.\frac{x^3}{3}\right|_0^h} \quad (5)\ (5)$$

$$= \frac{\frac{h^4}{4}}{\frac{h^3}{3}} = \frac{3h}{4}.$$

$\therefore$ The distance from the centre of the base $= h - \frac{3h}{4}$

$= \frac{h}{4}$ (5)

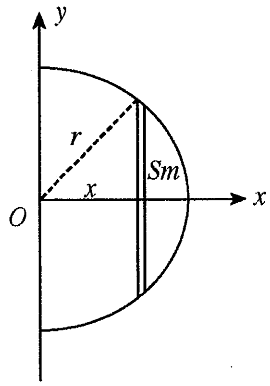
30

(i) Uniform solid hemisphere

By symmetry, the centre of mass lies on the x - axis. (5)

$Sm = \pi (r^2 - x^2)\,\delta x\,\sigma$,

where σ is the density

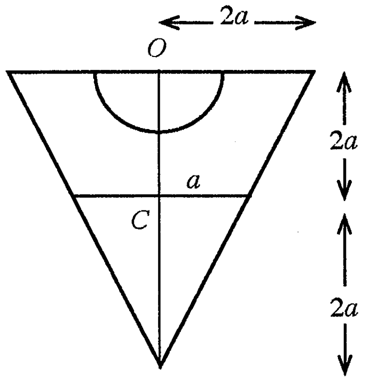


$$\bar{x} = \frac{\int_0^r \pi (r^2 - x^2)\,\sigma x\,dx}{\int_0^r \pi (r^2 - x^2)\,\sigma\,dx} \quad (5)\ (5)$$

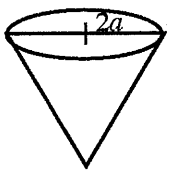
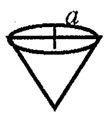
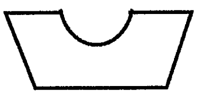
$$= \frac{\left.\left(\frac{r^2x^2}{2} - \frac{x^4}{4}\right)\right|_0^r}{\left.\left(r^2x - \frac{x^3}{3}\right)\right|_0^r} \quad (5)\ (5)$$

$$= \frac{\frac{r^4}{2} - \frac{r^4}{4}}{r^3 - \frac{r^3}{3}}$$

$$= \frac{3r}{8} \quad (5)$$



density σ

Object	Mass	Distance from O
	$\frac{16}{3}\pi a^3\rho$ (5)	a (5)
	$\frac{2}{3}\pi a^3\rho$ (5)	$\frac{5a}{2}$ (5)
	$\frac{2}{3}\pi a^3\rho$ (5)	$\frac{3a}{8}$ (5)
	$4\pi a^3\rho$ (5)	$\bar{x}$

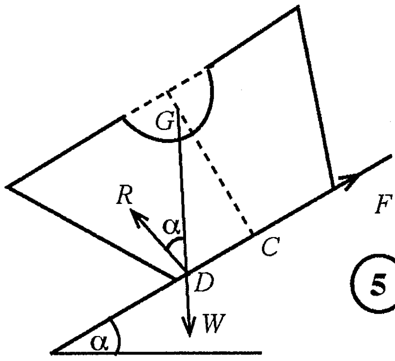
By symmetry, the centre of mass lies on the axis of symmetry. (5)

$$4\pi a^3\rho\,\bar{x} = \frac{16}{3}\pi a^3\rho a - \frac{2}{3}\pi a^3\rho\,\frac{5a}{2} - \frac{2}{3}\pi a^3\rho a\,\frac{3a}{8}$$ (20)

$$4\bar{x} = \frac{16}{3}a - \frac{5a}{2} - \frac{a}{4}$$ —

$$\bar{x} = \frac{41a}{48}$$ (5)

65



(5)

To prevent sliding

$\mu \geq \tan\alpha$ and so

$0.9 \geq \tan\alpha$ (10)

i.e. $\alpha \leq \tan^{-1}(0.9)$

To prevent rolling

$CD < a$ and so

$CG\tan\alpha < a$.

i.e. $\frac{55a}{48}\tan\alpha < a$ (10)

and so $\alpha < \tan^{-1}\left(\frac{48}{55}\right)$

25

17. (*a*) In a certain factory, machine A makes 50% of the items and the rest are made by machines B and C. It is known that 1%, 3% and 2% of the items made by A, B and C respectively are defective. The probability that a randomly selected item is defective is given to be 0.018. Find the percentages of items made by the machines B and C.

Given that a randomly selected item is defective, find the probability that it was made by the machine A.

(*b*) The time taken (in minutes) to travel to work from their homes of 100 employees of a certain factory are given in the following table:

Time taken	Number of employees
0 – 20	10
20 – 40	30
40 – 60	40
60 – 80	10
80 – 100	10

Estimate the mean, standard deviation and the mode of the distribution given above.

Later, all of the employees in the class interval 80 – 100 moved closer to the factory. It has changed the frequency of the class interval 80 – 100 from 10 to 0 and the frequency of the class interval 0 – 20 from 10 to 20.

Estimate the mean, standard deviation and the mode of the new distribution.

(*a*)

	A	B	C
Probability of Production	$\frac{1}{2}$	p	$\frac{1}{2} - p$
Probability of defects	$\frac{1}{100}$	$\frac{3}{100}$	$\frac{2}{100}$

D – randomly selected item is defective

$$P(D) = P(D/A)\,P(A) + P(D/B)\,P(B) + P(D/C)\,P(C)$$

$$0.018 = \frac{1}{100} \times \frac{1}{2} + \frac{3}{100} \times p + \frac{2}{100} \times \left(\frac{1}{2} - p\right) \quad (10)$$

$$3.6 = 1 + 6p + 2 - 4p$$

$$\Rightarrow p = 0.3 \quad (5)$$

$\therefore$ The percentage of items made by: machine B is 30% (5)

and machine C is 20% (5)

25

$$P(A/D) = \frac{P(D/A)\ P(A)}{P(D)} \quad (10)$$

$$= \frac{\frac{1}{100} \times \frac{1}{2}}{0.018} \quad (10)$$

$$= \frac{1}{100 \times 2}$$

$$= \frac{1}{\frac{18}{1000}}$$

$$= \frac{5}{18} \quad (5)$$

25

Time taken	f	Mid Point x (5)	$y = \frac{1}{10}x$	y^2	fy (5)	fy^2 (5)
0 - 20	10	10	1	1	10	10
20 - 40	30	30	3	9	90	270
40 - 60	40	50	5	25	200	1000
60 - 80	10	70	7	49	70	490
80 - 100	10	90	9	81	90	810
	100				$\sum fy = 460$ (5)	$\sum fy^2 = 2580$ (5)

$$\mu_y = \frac{\sum fy}{\sum f} = \frac{460}{100} = \frac{23}{5} \quad (5)$$

and

$$\sigma_y^2 = \frac{\sum fy^2}{\sum f} - \mu_y^2$$

$$= \frac{2580}{100} - \left(\frac{23}{5}\right)^2$$

$$= \frac{116}{25} \quad (5)$$

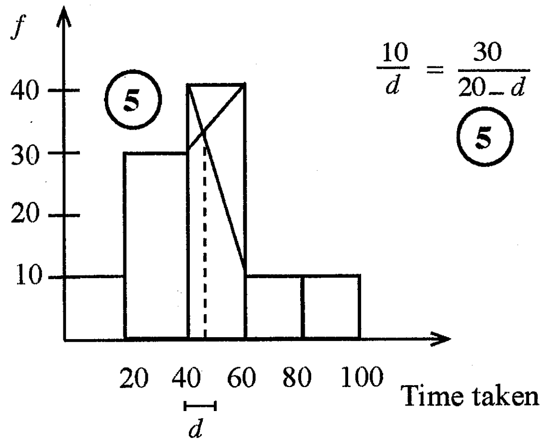
$$\therefore \ \sigma_y = \sqrt{\frac{116}{25}} \quad (5)$$

$$= \frac{2\sqrt{29}}{5}$$

$\therefore$ Mean $\mu_x = 10\mu_y = 10 \times \frac{23}{5} = 46$ (5)

$\therefore$ Standard deviation $\sigma_x = 10\sigma_y = 10 \times \frac{2\sqrt{29}}{5} = 4\sqrt{29} \approx 21.54$ (5)

Mode



$\frac{10}{d} = \frac{30}{20-d} \Rightarrow d = 5$ (5) $\therefore$ Mode $= 40 + d = 45$ (5)

65

(*b*) For the new distribution:

$$\mu_y = \frac{1}{100}\left[\sum_1^5 fy - f_1 y_1 - f_5 y_5 + 20 \times 1\right]$$

$$= \frac{1}{100}[460 - 10 - 90 + 20] = \frac{380}{100}$$ (5)

$$= \frac{19}{5}$$

$\therefore$ New mean $= 10 \times \frac{19}{5} = 38$ (5)

$$\sigma_y^2 = \left[\sum_1^5 fy^2 - f_1 y_1^2 - f_5 y_5^2 + 20 \times 1^2\right] - \left(\frac{19}{5}\right)^2$$

$$= \frac{1}{100}[2580 - 10 - 810 + 20] - \frac{361}{25}$$ (5)

$$= \frac{1780}{100} - \frac{361}{25}$$

$$= \frac{84}{25}$$

$\therefore \quad \sigma_y = \frac{\sqrt{84}}{5} = \frac{2\sqrt{21}}{5}$ (5)

$\therefore$ New Standard deviation $= 10 \times \frac{2\sqrt{21}}{5} = 4\sqrt{21} \approx 18.33$ (5)

Mode does not change (10) ($\because$ there is no change of the frequencies of the neighbourhood of the mode class)

35

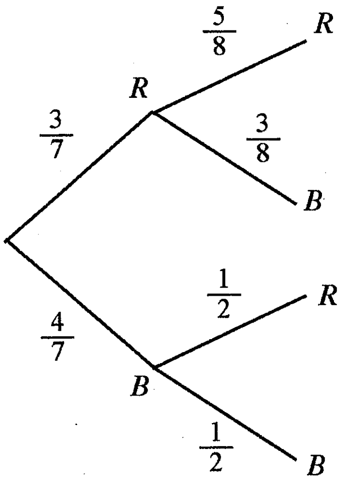
Old Syllabus

8. A bag A contains 3 red balls and 4 black balls, and another bag B contains 4 red balls and 3 black balls. The balls in bag A and bag B are identical in all aspects except for their colour. A ball is drawn at random from bag A and put into bag B. Now, a ball is drawn at random from bag B. Find the probability that

(i) the ball drawn from bag B is black,

(ii) the ball drawn from bag B is black, given that the ball drawn from bag A is red.

A	B
3 Red 4 Black	4 Red 3 Black



(i) $P(\text{Ball from } B \text{ is black}) = \frac{3}{7} \times \frac{3}{8} + \frac{4}{7} \times \frac{1}{2} = \frac{9}{56} + \frac{16}{56} = \frac{25}{56}$ (5) (5) (5)

(ii) $P(\text{Black from } B \,|\, \text{red from } A) = \dfrac{P(\text{Black from } B \text{ and red from } A)}{P(\text{red from } A)}$

$$= \frac{\frac{3}{7} \times \frac{3}{8}}{\frac{3}{7}}$$

$$= \frac{3}{8}$$ (10)

(Or just from the branch from the tree.)

25

10. The mean and the standard deviation of marks obtained by students of a class for a question paper in statistics are 40 and 15, respectively. These marks were transformed using the formula $t = \frac{1}{3}(70 + 2x)$, where x is the original mark. Find the mean and the standard deviation of the transformed marks. The median of the transformed marks is 55. Find the median of the original marks.

$$\mu_t = \frac{1}{3}(70 + 2\mu_0) = \frac{1}{3}(70 + 80) = 50 \quad (5)$$

(5)

$$\sigma_t = \frac{2}{3}\,\sigma_0 = \frac{2}{3} \times 15 = 10 \quad (5)$$

$$M_t = \frac{1}{3}(70 + 2M_0) \quad (5)$$

$$55 = \frac{1}{3}(70 + 2M_0)$$

$$M_0 = \frac{95}{2} = 47.5 \quad (5)$$

25